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Some Properties of Geometric LINNIK & Generalized Geometric LINNIK Distributions

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ABSTRACT: Consider the geometric Linnik distribution GL(α,λ) with characteristic function $\phi(t)=\dfrac{1}{1+ln(1+\lambda|t|^{\alpha})}, \quad \lambda>0,\ 0<\alpha\le 2.$ and type II Generalized Geometric Linnik distribution $GeGL_2(\alpha,\lambda,\tau)$ with $\phi(t)=\left[\dfrac{1}{1+ln(1+\lambda|t|^{\alpha})}\right]^{\tau}, t\in R, 0<\alpha\le 2, \lambda,\tau>0.$

In this paper Infinte divisibility property of geometric Linnik and generalized geometric Linnik distribution is derived. A representation of geometric Linnik random variables in terms of geometric exponential and stable random variables is presented. Some more properties of geometric Linnik distribution is obtained .

KEY WORDS: geometric Linnik distribution, generalized geometric Linnik distribution, Infinte divisibility,

1.INTRODUCTION

[6] proved that the function $\phi(t)=\dfrac{1}{1+\lambda|t|^{\alpha}}, 0<\alpha\le 2, \lambda>0$ (1.1)

is the characteristic function of a probability distribution. The distribution corresponding to the characteristic function (1.1) is called Linnik (or α - Laplace) distribution. A random variable X with characteristic function ϕ in (1.1) is denoted by $X \underset{=}{d} L(\alpha,\lambda)$. Note that the $L(\alpha,\lambda)$ distributions are symmetric and for $\alpha=2$, it becomes the classical symmetric Laplace distribution. As a generalization of the Linnik distribution [8] introduced semi α -Laplace distribution.

A random variable X on R has semi α -Laplace distribution if its characteristic function $\phi(t)$ is of the form

$$\phi(t) = \frac{1}{1+|t|^{\alpha}\delta(t)} \tag{1.2}$$

where $\delta(t)$ satisfies the functional equation $\delta(t) = \delta\left(p^{1/\alpha}t\right), 0<p<1, 0<\alpha\le 2.$ (1.3)

[7] introduced generalized Linnik law with characteristic function $\phi(t) = \dfrac{1}{\left(1+|t|^{\alpha}\right)^{\nu}}, \quad \nu>0, \quad 0<\alpha\le 2.$ (1.4)

[5] developed Pakes generalized Linnik first order autoregressive process and studied its properties. [11] studied about geometric Linnik distribution.Some properties of type I and type II generalized geometric Linnik distribution were studied in [12].Timeseries models with generalized geometric Linnik marginal were developed in [13].

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DEFINITION 1.1

A random variable X on R is said to have geometric Linnik distribution and write $X \underset{=}{d} GL(\alpha,\lambda)$ if its characteristic function $\phi(t)$ is

$$\phi(t) \quad = \quad \frac{1}{1+\ln(1+\lambda|t|^{\alpha})}, t \in R, 0<\alpha \leq 2, \lambda>0. \qquad (1.5)$$

DEFINITION 1.2

A random variable X on R is said to have type I generalized geometric Linnik distribution and write $X \underset{=}{d} GeGL_1(\alpha,\lambda,p)$ if it has the characteristic function

$$\phi(t)=\frac{1}{1+p\ln(1+\lambda|t|^{\alpha})},\ 0<\alpha \leq 2,\ p>0,\ \lambda>0. \qquad (1.6)$$

DEFINITION 1.3

A random variable X on R has type II generalized geometric Linnik distribution and writes $X \underset{=}{d} Ge\,GL_2(\alpha,\lambda,\tau)$, if it has the characteristic function

$$\phi(t)=\left[\frac{1}{1+ln(1+\lambda|t|^{\alpha})}\right]^{\tau}, t \in R, 0<\alpha \leq 2, \lambda,\tau>0. \qquad (1.7)$$

Note that when $\tau=1$, type II ggeneralized geometric Linnik distribution reduces to geometric Linnik distribution.

DEFINITION 1.4

A random variable X is said to be infinite divisible if for every positive integer n,there exist independently and identically distributed random variables $X_{n,1}$, $X_{n,2,\ldots..}$, $X_{n,n}$ such that X is distributed as $X_{n,1+}$ $X_{n,2}$+…..+ $X_{n,n}$

Equivalently, a characteristic function ϕ is said to be infiniety divisible if for every positive integer n, there exists a characteristic function ϕ_n such that $\phi(t)=(\phi_n(t))^n$.

It is known that the class of infinitely divisible distributions coincides with the class of limit distributions of the row sums of certain triangular arrays. Note that the class of all infinite divisible distributions coincides with the class of all continuous convolution semi groups. For the properties of infinite divisibility, see [4].

[3] introduced the concept of geometric infinite divisibility of random variables.

DEFINITION 1.5

A random variable Y is said to be geometrically infinitely divisible, if Y can be expressed as

$Y \underset{=}{d} \sum_{j=1}^{Np} X_p^{(j)}$ for every $p \in (0,1)$ where $P(Np=k)=pq^{k-1}, k=1,2,\ldots$ and N_p and $X_p^{(j)}$ $(j=1,2,\ldots)$ are independent and q=1-p.

[9] proved that every geometrically infinitely divisible distribution is infinitely divisible.

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II.SOME PROPERTIES OF GEOMETRIC LINNIK &GENERALIZED GEOMETRIC LINNIK DISTRIBUTIONS

THEOREM 2.1
$GL(\alpha,\lambda)$ distribution is infinitely divisible .

PROOF

For $GL(\alpha,\lambda)$ distribution, $\phi(t) = \frac{1}{1+\ln(1+\lambda|t|^{\alpha})}$,

$$e^{1-\frac{1}{\phi(t)}} = \frac{1}{1+\lambda|t|^{\alpha}}. \quad (2.1)$$

But $\frac{1}{1+\lambda|t|^{\alpha}}$ is the characteristic function an infinitely divisible distribution. Thus the distribution with characteristic function $\frac{1}{1+\ln(1+\lambda|t|^{\alpha})}$, $\lambda > 0, 0 < \alpha \le 2$ is the characteristic function of a geometrically infinitely divisible distribution. By [10] every geometrically infinitely divisible distribution is infinitely divisible.

THEOREM 2.2
$GeGL_2(\alpha,\lambda,\tau)$ distributions are infinitely divisible.

PROOF

Follows from Theorem 2.1
[9] introduced geometric exponential distribution while studying the geometric infinite divisibility of harmonic mixtures of random variables.

DEFINITION 2.1
A random variable X on $[0,\infty)$ is said to have geometric exponential distribution if it has the Laplace transform

$$\phi(\delta) = E\left(e^{-\delta X}\right) = \frac{1}{1+1n(1+\delta)}, \quad \delta > 0. \quad (2..2)$$

A representation of geometric Linnik random variables in terms of geometric exponential and stable random variables is presented below.

THEOREM 2.3
Let X and Y be independent random variables such that X has geometric exponential distribution with Laplace transform $\frac{1}{1+1n(1+\delta)}$ and Y is stable with characteristic function $e^{-\lambda|t|^{\alpha}}$, $0 < \alpha \le 2$. Then $X^{1/\alpha}Y \stackrel{d}{=} GL(\alpha,\lambda)$.

PROOF

$$\phi_{X^{1/\alpha}Y}(t) = E\left(e^{itX^{1/\alpha}Y}\right) = \int_0^{\infty} E\left[e^{itX^{1/\alpha}Y} \big/ X = x\right] dF(x)$$

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$$= \int_0^\infty \phi_Y(tx^{1/\alpha})dF(x)$$

$$= \frac{1}{1+1n\left(1+\lambda|t|^\alpha\right)}.$$

DEFINITION 2.2

A random variable X on R has the generalized Linnik distribution and write $X \underset{=}{d} GeL(\alpha,\lambda,p)$ if it has the characteristic function $\phi(t) = \frac{1}{(1+\lambda|t|^\alpha)^p}$, $p>0, \lambda>0, 0<\alpha\le 2.$ (2..3)

[1]&[2] considered a generalization of $GeL\ (\alpha,\lambda,p)$ distribution and studied its properties. They discussed the analytic and asymptotic properties of this distribution and obtained some integral and series representation of its probability density.

THEOREM 2.4

Let $X_1, X_2,...$ be independent and identically distributed $GL(\alpha,\lambda)$ random variables and N_p be geometric with mean $\frac{1}{p}$. Define $Y = X_1 + X_2 + ... + X_{N_p}$ where N_p is independent of X_i's. Then $Y \underset{=}{d}\ Ge\,GL_1(\alpha,\lambda,1/p)$.

PROOF follows easily.

THEOREM 2.5

1) Let $X_1, X_2,...$ be independent and identically distributed $GeL(\alpha,\lambda,1/n)$ random variables. Let N be geometric with mean n independent of X_i's. Then $X_1 + X_2 + ... + X_N$ converge in distribution to Z where $Z \underset{=}{d} GL(\alpha,\lambda)$.

PROOF

Consider $\frac{1}{(1+\lambda|t|^\alpha)^{1/n}} = \frac{1}{1+\left[(1+\lambda|t|^\alpha)^{1/n}-1\right]}$. Since Linnik distribution is infinitely divisible, $\frac{1}{(1+\lambda|t|^\alpha)^{1/n}}$ is the characteristic function of a probability distribution.

Let $$\phi_n(t) = \frac{1}{1+n\left[(1+\lambda|t|^\alpha)^{1/n}-1\right]}.$$

Then

$$\phi(t) = \lim_{n\to\infty} \phi_n(t)$$

$$= \frac{1}{1+\lim_{n\to\infty} n\left[(1+\lambda|t|^\alpha)^{1/n}-1\right]} = \frac{1}{1+ln(1+\lambda|t|^\alpha)}.$$

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THEOREM 2.6

The function $\phi(t)=\dfrac{1}{1+ln(1+\lambda|t|^{\alpha})}$ on R is a characteristic function if and only if $\alpha\in(0,2]$.

PROOF

Suppose for some $\alpha>0$, the function $\phi(t)$ is a characteristic function. Then we have to prove that $\alpha\in(0,2]$. The case $\alpha<0$ is impossible due to the requirement that $\lim_{t\to 0}\phi(t)=1$ for the characteristic function ϕ.

Note that for each positive integer n, the function $\phi_n(t)=\left[\dfrac{1}{1+\frac{1}{n}ln(1+\lambda|t|^{\alpha})}\right]^n$ is also a characteristic function. Let F_n denote the distribution function with characteristic function ϕ_n. Then F_n converges weakly to a Linnik characteristic function $\phi(t)=\dfrac{1}{1+\lambda|t|^{\alpha}}$. This implies that $\alpha\in(0,2]$.

For fixed $\alpha\in(0,2]$, the function $\dfrac{1}{1+\lambda|t|^{\alpha}}$ is the characteristic function of Linnik distribution. By Theorem 2.5, $\dfrac{1}{1+ln(1+\lambda|t|^{\alpha})}$ is a characteristic function.

THEOREM 2.7

$GL(\alpha,\lambda)$ is normally attracted to stable law.

PROOF

Let $S_n=X_1+X_2+...+X_n$ where $X_i's$ are independent and identically distributed $GL(\alpha,\lambda)$ random variables. The characteristic function of $n^{-1/\alpha}S_n$ is

$$\phi_{n^{-1/\alpha}S_n}(t)=\left[\frac{1}{1+ln(1+\frac{\lambda|t|^{\alpha}}{n})}\right]^n$$

$$=\left[\frac{1}{1+\frac{\lambda|t|^{\alpha}}{n}+o(\frac{1}{n^2})}\right]^n\to e^{-\lambda|t|^{\alpha}},\text{as } n\to\infty.$$

This completes the proof.

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