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A Study of Properties of Euler's Zeta Function by using Riemann Hypothesis

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ABSTRACT: The aim of this paper to study the zeros of the zeta function and the relative properties of the function. The zeta function has two types of zeros, First trivial zeros that consist of all negative even integers. Second, an infinite number of non-trivial zeros which are all complex. In this study, we will first introduce a general framework for spatial data mining which takes into account the mentioned characteristics of spatial data. We can show that this approach allows a tight and efficient integration of spatial data mining algorithms along with spatial database systems. We will further present algorithms for the tasks of spatial clustering, spatial characterization, spatial trend detection and spatial classification utilizing the proposed framework. Furthermore, an example application is discussed for such algorithms.

I. INTRODUCTION

Any real number, say 's' greater than 1, the Euler's zeta function is defined as the infinite sum as following

$$\zeta(s) = 1+\frac{1}{n^s} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .(1)$$

Where n is natural number and s is any bigger number than 1 [8]. First we are going to look the derived results from Euler's function. Then we will constitute the model which is useful result in meta-querying to access the discovered information from giant database.
Let us start the thing by deriving the Harmonic series simply given as following

$$1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/5 + 1/7 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (2)$$

Now, Can we guess the result of sum of the above Harmonic series? As the numbers of terms in the series is although infinite but the range of sum for the solution is rather finite. This result can be probably estimated with some limit value. This feature we will use as indexes described. The indexes further helps in the algorithm to find the data in the complex database.
We can apply the Euler's theorem further by dividing the above defined harmonic series in as sum of prime harmonic terms and some other terms shown as following

$$S_1 = 1+ 1/2 + 1/3 + 1/5 +1/7 + 1/11 + 1/13 + 1/17 + 1/19 +\ldots\ldots\ldots (3)$$

Other remaining terms are below.

$$S_2 = 1/4 + 1/6 + 1/8 + 1/9 + 1/10 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \qquad (4)$$

The value of prime harmonic series of equation (3) can be directly evaluated from Euler's function and the same can be done in multiple fashions for equation (4) For an example a lot of terms of equation (4) can be re-written as following.

$$1/4 + 1/6 + 1/8 + 1/10 \ldots = (1/2) \times (1/2 + 1/3 + 1/4 + 1/5 \ldots) \qquad (5)$$

The idea is to divide the huge infinite series into few series of "Prime Harmonics", which can be very easily evaluated by Euler's formulae [3-6]. The similar efforts we can do in computing the parameters like indexes, supports, weights, range etc. in making discovery in spatial databases. The found pattern can be used by neural networks in order to do the search efficient and faster with a lot of ease.

II. MORE ABOUT ZETA FUNCTION

Euler defined the zeta function and then he showed that it has a deep and profound connection with the pattern of the primes. We are providing here the definitions of Zeta function given by Euler

$$\zeta(s) = \pi\left(\frac{1}{1-n^s}\right) \qquad \ldots\ldots\ldots \qquad (6)$$

Where n are the prime numbers and s is any real number greater than 1.

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Then above definition indicates that the Euler's function can be observed as multiple of prime terms. Now we are focusing the result derived from the above study like Euler's discovery to write the following formulae

$$(1-x)^{-1} = 1/(1-x)$$
$$= 1+x+x^2+x^3+x^4+ \ldots\ldots\ldots\ldots \quad (7)$$

So, we can write,

$$\frac{1}{1-p^s} = 1+1/p^s+1/p^{2s}+1/p^{3s} \quad \ldots\ldots\ldots\ldots \quad (8)$$

This expression on the left is a typical term in Euler's infinite product, of course, so the above equation provides an infinite sum expression for each term in the infinite product. What Euler did next was multiply together all of these infinite sums to give an alternative expression for his infinite product. Using the ordinary algebraic rules for multiplying (a finite number of finite) sums, but applying them this time to an infinite number of infinite sums, you see that when you write out the right-hand side as a single infinite sum, its terms are all the expressions of the product form.[1]
Now, from the point of view of the subsequent development of mathematics it was not so much the fact that the prime harmonic series has an infinite sum that is important, even though it did provide a completely new proof of Euclid's result that there are infinitely many primes [1, 4]. Rather, Euler's infinite product formula for $\zeta(s)$ marked the beginning of analytic number theory.
The French mathematician Lejeune Dirichlet generalized Euler's method to prove that in any arithmetic progression a, $a+k$, $a+2k$, $a+3k$, . . ., where a and k have no common factor, there are infinitely many primes. Euclid's theorem can be regarded as the special case of this for the arithmetic progression 1, 3, 5, 7, ... of all odd numbers. The principal modification to Euler's method that Dirichlet made was to modify the zeta function so that the primes were separated into separate categories, depending on the remainder they left when divided by k. His modified zeta function had the form as defined below

$$L(s,\chi) = \chi(1)/1^s + \chi(2)/2^s + \chi(3)/3^s + \ldots\ldots\ldots\ldots \quad (9)$$

where $\chi(n)$ is a special kind of function which Dirichlet called a "characteristic function" that splits the primes up in the required way. In particular, it must be the case that $\chi(mn) = \chi(m)\chi(n)$ for any m, n. The other conditions are that $\chi(n)$ depends only on the remainder you get when you divide n by k, and that $\chi(n) = 0$ if n and k have a common factor [1, 2].

III. ZEROS OF ZETA FUNCTIONS

Riemann pointed out that the zeta function defined by Euler has zeros of two types. The first types are trivial zeros that consist of all negative even integers and the second types are an infinite number of non-trivial zeros which are all complex, and are known to lie in the strip 0 < ® (s) < 1. The trivial zeros are well understood, but the study of the non-trivial zeros of zeta is still ongoing.
The Riemann zeta function can be given in various equivalent forms. In the right half-plane σ > 0, the zeta function can be defined as follows:

$$2\,\mathrm{SIN}(\pi s)\,\pi(s-1)\,\zeta(s) = \int \frac{(-x)^{s-1}}{e^x-1}\,dx \quad \text{---------} \quad (10)$$

This formula gives a good approximation for the zeta function in the critical strip, and can thus be used in the study of the Riemann Hypothesis. The uniqueness obtained from the principle of analytic continuation guarantees that this definition is consistent with equation (2).
The Zeta function satisfies an important reflexive property. This property we will use in mining data and information and to control duplicity in the databases. We see that mathematicians study the zeta function and the Riemann Hypothesis from various perspectives and fields of mathematics such as number theory, complex analysis, and random matrices. Here, we would like to introduce a new approach which simplifies the study of Riemann Hypothesis from the complex plane to a real line[9, 12]. This can be done by two methods. First, by fixing the imaginary part of arbitrary zeros of the zeta function, and thus studying the real part of those zeros along a horizontal line. Second, we can study the imaginary part of the zeros of zeta on the critical line, and search for embedded patterns and relations amongst them.
This provides valuable information on the distribution of the zeros of the zeta function on the critical line. We have listed the following Riemann Hypothesis formulae generated as data mining tools.

- Correlation Coefficient (CC) = $\frac{(n-1)\sum_{i=1}^{n}(p_i-\bar{p})(a_i-\bar{a})}{[\sum_{i=1}^{n}(p_i-\bar{p})^2][\sum_{i=1}^{n}(a_i-\bar{a})^2]}$ (F 1)
- Mean Absolute Error (MAE) = $\frac{\sum_{i=1}^{n}|p_i-a_i|}{n}$ (F 2)
- Root Mean –Squared Error (RMSE) = $\sqrt{\frac{\sum_{i=1}^{n}(p_i-a_i)^2}{n}}$ (F 3)

- Relative Absolute Error (RAE) = $\frac{\sum_{i=1}^{n}|p_i - a_i|}{\sum_{i=1}^{n}|a_i - \bar{a}|}$ (F 4)
- Root Relative Squared Error (RRSE) = $\sqrt{\frac{\sum_{i=1}^{n}(p_i - a_i)^2}{\sum_{i=1}^{n}(a_i - \bar{a})^2}}$ (F 5)

The non-trivial zeros of the Riemann zeta function lie on the critical line <(s) or = 1. Riemann first stated this hypothesis in his paper "On the Number of Prime Numbers less than a Given Quantity" that was published in 1859. Mathematicians have since been struggling to prove the hypothesis. This includes Riemann himself, who admits to having given up, at least temporarily, on a proof of Riemann hypothesis after several unsuccessful attempts [7-11]. Riemann conjectured that all the non-trivial zeros of zeta lie on this critical line. This is known as the Riemann Hypothesis , considered to be one of the most important open problems in mathematics.

Since all the non-trivial zeros of the zeta function calculated thus far have real part equal to ½, it suffices to only study their imaginary parts. We study the zeros of zeta on the critical line by studying the sequence given by the following definition. It is important to note that by n, we mean a positive integer, and that we use the notation $\{\alpha_n\}_n$ for countable sequences of real numbers α_n .

Definition: For n covering all positive integers, denote by $\{t_n\}_n$ the sequence of the positive imaginary parts of all the zeros of the zeta function on the critical line in the upper half of the complex plane, in increasing order.

We wish to find relationships between a given element t_n, and previous elements $t_1, t_2, t_3, \ldots , t_{n-1}$. The first problem to answer at this point is what type of relationship we are seeking. We search for possible formulas for t_n that are linear in all or some of $t_1, t_2, t_3, \ldots , t_{n-1}$. Unfortunately, no good model could be found. This is expected; for it is generally believed that the zeros of the zeta function in particular, and any L-function in general, are linearly independent [4,5]. In order to get past this problem, we chose a different approach for a solution. Instead of studying the sequence $\{t_n\}_n$, We map the sequence to another sequence $\{f(t_n)\}_n$, and then study this new sequence for possible linear patterns.

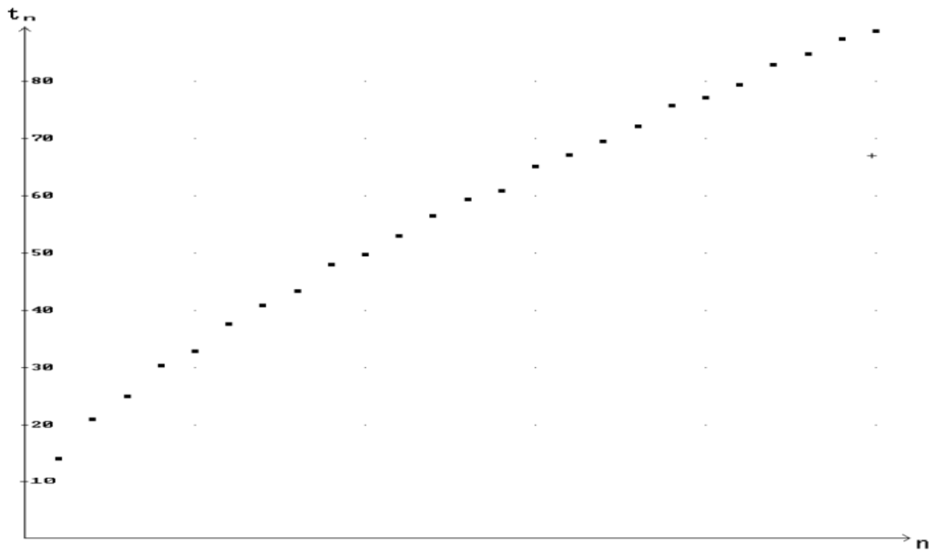

Fig. : The complex parts of the first thirty non-trivial zeros of zeta

Based on mathematical intuition, accompanied with empirical trial and error, we identified many possibilities for mapping functions f.

IV. PROPERTIES OF ZETA FUNCTION

The development of the models was done in Weka; a collection of machine learning algorithms for data mining tasks. We give examples of some of the suggested models and compare them. As a measure of success of any model, we use the statistical measures Correlation Coefficient, Mean Absolute Error, Root Mean –Squared Error, Relative Absolute Error and Root Relative Squared Error as defined above. We would like to note that there is no specific rationale behind the choice of the statistical measures used; except that they give good point-wise and overall statistics on the validity and success of the models. The statistical measures Root Mean Square Error and Mean Absolute Error are concerned with

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the average difference errors. The Relative Absolute Error and Root Relative Squared Error statistics give more intuition about the difference errors relative to the actual values studied. The latter statistics might be misleading. Take for example an error of 1% in a calculation where the actual outcome is known to be 100. The approximate answer given is 100 ± 1.[6, 10]

On the other hand, say we approximate a function with smaller relative error 0.5%, but are approximating a value that is actually 10^6. The estimated answer would be off by $\pm 10^6 \times 0.5\% = \pm 5 \times 10^3$. Although the relative error is smaller in the second example, the difference error (which is the mean absolute error for data of size one) is much smaller in the first case. Therefore, to obtain strong local approximations, we attempt to minimize the mean absolute and the root mean square errors.

For practicality in the speed of the study, we calculated all our models and their statistical measures of accuracy using the first 10^4 elements of $\{t_n\}_n$, split into two subsets: The first 5×10^3 elements used as training data, while the rest being test data.

Table 1: Sample of input data for the model of our study say, $n\frac{t_{n+1}}{t_n}$.

n	$n\frac{t_{n+1}}{t_n}$
5	5.706104125321350
50	51.009398160818996
500	500.978044956544980
2000	2001.021407482380000
8000	8000.783730290239000
10000	10000.882909059201000

For each study, we examined the relationship between the proposed model $\{f(t_n)\}_n$ and the number of the instance n. Tables 1 and 2 show samples of the training and test instances for two of the proposed models [4-7]. List of the models that were proposed, their approximate linear models, and their statistical measures of validity. The models were trained and tested through Weka [3].

The mapping model $n\frac{\ln(t_{n+1})}{\ln(t_n)}$ has the best overall statistical measures, and thus has the best linear approximation. Then it shows a sample of terms of the sequence $\{t_n\}_n$ along with their corresponding approximations, an + b, calculated using this model, with the sample inputs.

Table 2: Sample of input data for the model $n\frac{\ln(t_{n+1})}{\ln(t_n)}$.

n	$n\frac{\ln(t_{n+1})}{\ln(t_n)}$
5	5.189007239929700
50	50.201333522284500
500	500.145867031175040
2000	2000.130412290700200
8000	8000.087023205120200
10000	10000.095984553700000

Random actual values from $\{t_n\}$ n and their approximations given by the model

$n\frac{\ln(t_{n+1})}{\ln(t_n)}$ choice of b = .12244.

The remainder of the study is devoted to this model since it has the best performance. Having identified a satisfactory model for the sequence $\{t_n\}_n$, we turn to the mathematical analysis process of our study.[11, 12] In the next section of this chapter validates the model mathematically. It gives us the interesting idea to utilize the Zeta and Riemann Hypothesis based Zeta function in doing data mining.

V. CONCLUSION

Euler thought of splitting this sum into two parts, a sum of all the terms involving the primes, and a sum involving the terms with composite numbers. He wanted to show that the latter sum is convergent, and thus conclude that the sum of the reciprocals of all primes diverges. Yet, since it is infinite, and its two parts do not both converge, Euler was not able to split the harmonic series the way he wanted. That is, an infinite series cannot be split into various parts unless all the parts converge. This series converges as long as the single complex variable s is strictly larger than one. Hence, it can be split into the two sums the way Euler desired. The zeta function also appears in physics; especially in areas relevant to chaos in classical and quantum mechanics. Here in our proposed research we are using to retrieve data and information from complex pattern like special databases and Data ware House. The discovery process for spatial data is more complex than for relational data. This applies to both the efficiency of algorithms as well to the complexity of possible patterns that can be found in a spatial database.

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