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Common Fixed Points in Hilbert Spaces

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ABSTRACT: The results proved in this paper regarding common fixed point theorems using concepts of compatibility, weak compatibility and commutativity improve the results of Imdad and Ali [4] and Gupta and Badshah [3] in Hilbert space. Our main results are useful and are different in comparison to previously proven results in the theory of fixed points.

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KEYWORDS: Common fixed point, Hilbert space, closed subset, compatible mappings and weakly compatible mappings.

I. INTRODUCTION

In the literature of fixed-point theory, vast amount of fixed point theorems were proved satisfying certain contractive type conditions. The notion of generalized contraction mappings was introduced by Ciric [1]. Das and Gupta [2] introduced fixed point theorems in reflexive Banach spaces. Most of the fixed point theorems in metric spaces satisfying contraction conditions may be extended to the abstract spaces such as Hilbert spaces, Banach spaces and locally convex spaces etc. Rhoades [9, 10] provided comprehensive survey on various definitions of contractive mappings.

Jungck [5] proved common fixed point theorem for commuting mappings which generalized the Banach's fixed point theorem. Sessa [11] generalized the notion of commuting mappings by defining weakly commuting mappings. Jungck [6] generalized the concept of weakly commuting mappings by defining compatible maps which asserts that a pair of self-maps (S, T) is said to be compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, whenever $\{ x_n \}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$. Later on, Jungck [7] generalized the notion of compatibility by introducing the concept of weak compatibility which asserts that S and T are said to be weakly compatible if they commute at their coincidence points that is if Su = Tu for some $u \in X$, then STu = TSu. A pair of compatible maps is weakly compatible but the converse is not true in general.

Imdad and Ali [4] observed that the fixed point theorem proved by Nigam et al. [8] in Hilbert space remains true in metric space and obtained a slightly improved form of their result in metric space which is as follows:

Theorem 1.1. [4] Let T_1 and T_2 be self-mappings of a closed subset C of complete metric space X satisfying

$$\|T_1x - T_2y\|^2 \le a\|x-y\|^2 + b\|x - T_1x\|\|y - T_2y\| + c\|x - T_2y\|\|T_1x - y\|$$

for all $x, y \in C$, and a,b,c non negative reals, with max {a+b, a+c} <1.

Then T_1 and T_2 have a unique common fixed point.

Gupta and Badshah [3] proved the following results to obtain fixed point theorems for two mappings using functional inequality in Hilbert space.

Theorem 1.2. [3] Let C be any closed subset of a Hilbert space H and S, T: C→C be two mappings those satisfy

$$\|Sx - Ty\| \le \alpha \frac{\|y - Ty\|[1 + \|x - Sx\|]}{1 + \|x - y\|} + \beta\|x - y\|$$

for each $x, y \in C, x \neq y, 0 < \alpha, \beta < 1$ and $\alpha + \beta < 1$. Then there exists a unique common fixed point of S and T.

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Theorem 1.3. [3] Let C be any closed subset of a Hilbert space H and S, T : C→C be two mappings those satisfy

$$\|Sx - Ty\| \le \alpha\left[\frac{\|x - Sx\|^3 + \|y - Ty\|^3}{\|x - Sx\|^2 + \|y - Ty\|^2}\right] + \beta[\|x - Sx\| + \|y - Ty\|] + \gamma\|x - y\|$$

for each $x, y \in C, x \neq y, 0 < \alpha, \beta, \gamma < 1$ and $2\alpha + 2\beta + \gamma < 1$. Then there exists a unique common fixed point of S and T.

II. MAIN RESULT

Motivated by Imdad and Ali [4] and Gupta and Badshah [3], we presume yet another extension of their results by proving a unique common fixed point theorem using notions of compatibility, weak compatibility, commutativity and increasing the number of mappings from two to six in Hilbert space.

Theorem 2.1. Let X be a closed subset of a Hilbert space H and let A, B, S, T, P and Q be mappings from X into itself satisfying AB(X) ⊂ Q(X), ST(X) ⊂ P(X) and for each x, y in X
(2.1.1)

$$\|ABx - STy\| \le \alpha\left[\frac{\|Px - ABx\|^3 + \|Qy - STy\|^3}{\|Px - ABx\|^2 + \|Qy - STy\|^2}\right]$$
$$+\beta\left[\frac{\|Px - ABx\|^2 + \|Qy - STy\|^2}{\|Px - ABx\| + \|Qy - STy\|}\right]$$
$$+\gamma\|Px - Qy\| + \delta[\|Px - STy\| + \|ABx - Qy\|]$$

where $\|Px - ABx\| + \|Qy - STy\| \neq 0$ and $\alpha, \beta, \gamma, \delta \ge 0$ with $2\alpha + 2\beta + \gamma + 2\delta < 1$.
If (P, AB) is compatible, P or AB is continuous and
(Q, ST) is weakly compatible then AB, ST, P and Q have a unique common fixed point. Furthermore, if the pair (A, B), (S, T), (P, A) and (Q, S) are commuting then A, B, S, T, P and Q have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X.
As AB(X) ⊂ Q(X) and ST(X) ⊂ P(X), then there exists $x_1, x_2 \in X$ such that $ABx_0 = Qx_1 = y_0$ and $STx_1 = Px_2 = y_1$. We can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that
$y_{2n} = Qx_{2n+1} = ABx_{2n}$ and $y_{2n+1} = STx_{2n+1} = Px_{2n+2}$
for n = 0, 1, 2.….

$$\|y_{2n+2} - y_{2n+1}\| = \|ABx_{2n+2} - STx_{2n+1}\|$$
$$\le \alpha\left[\frac{\|Px_{2n+2} - ABx_{2n+2}\|^3 + \|Qx_{2n+1} - STx_{2n+1}\|^3}{\|Px_{2n+2} - ABx_{2n+2}\|^2 + \|Qx_{2n+1} - STx_{2n+1}\|^2}\right]$$
$$+\beta\left[\frac{\|Px_{2n+2} - ABx_{2n+2}\|^2 + \|Qx_{2n+1} - STx_{2n+1}\|^2}{\|Px_{2n+2} - ABx_{2n+2}\| + \|Qx_{2n+1} - STx_{2n+1}\|}\right]$$
$$+\gamma\|Px_{2n+2} - Qx_{2n+1}\| + \delta[\|Px_{2n+2} - STx_{2n+1}\| + \|ABx_{2n+2} - Qx_{2n+1}\|].$$
$$\le \alpha[\|Px_{2n+2} - ABx_{2n+2}\| + \|Qx_{2n+1} - STx_{2n+1}\|]$$
$$+\beta[\|Px_{2n+2} - ABx_{2n+2}\| + \|Qx_{2n+1} - STx_{2n+1}\|]$$
$$+\gamma\|Px_{2n+2} - Qx_{2n+1}\| + \delta[\|Px_{2n+2} - STx_{2n+1}\| + \|ABx_{2n+2} - Qx_{2n+1}\|].$$
$$= \alpha[\|y_{2n+1} - y_{2n+2}\| + \|y_{2n} - y_{2n+1}\|]$$
$$+\beta[\|y_{2n+1} - y_{2n+2}\| + \|y_{2n} - y_{2n+1}\|]$$
$$+\gamma\|y_{2n+1} - y_{2n}\| + \delta[\|y_{2n+1} - y_{2n+1}\| + \|y_{2n+2} - y_{2n}\|]$$
$$\le \alpha[\|y_{2n+1} - y_{2n+2}\| + \|y_{2n} - y_{2n+1}\|]$$
$$+\beta[\|y_{2n+1} - y_{2n+2}\| + \|y_{2n} - y_{2n+1}\|]$$
$$+\gamma\|y_{2n+1} - y_{2n}\| + \delta[\|y_{2n+2} - y_{2n+1}\| + \|y_{2n+1} - y_{2n}\|].$$
$$\le \frac{(\alpha + \beta + \gamma + \delta)}{(1 - \alpha - \beta - \delta)}\|y_{2n} - y_{2n+1}\|.$$

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$\|y_{2n+2} - y_{2n+1}\| \leq k\|y_{2n} - y_{2n+1}\|$ where $k = \frac{(\alpha+\beta+\gamma+\delta)}{(1-\alpha-\beta-\delta)} < 1$

Thus for all n, we have

$$\|y_{n+1} - y_n\| \leq k\|y_n - y_{n-1}\|$$
$$\|y_{n+1} - y_n\| \leq k^n\|y_1 - y_0\|$$

Thus, $\{y_n\}$ is a Cauchy sequence in X and X is closed subset of a Hilbert space H, there exist a point u in X such that $\{y_n\} \to u$. Consequently, the subsequences $\{ABx_{2n}\}$, $\{Qx_{2n+1}\}$, $\{STx_{2n+1}\}$ and $\{Px_{2n+2}\}$ of sequence $\{y_n\}$ also converges to u in X.

Case I. Suppose P is continuous, we have
$P^2x_{2n} \to Pu$ and $P(AB)x_{2n} \to Pu$.
The compatibility of the pair (P, AB) gives that
$AB(P)x_{2n} \to Pu$.

Step 1. Putting $x = Px_{2n}$ and $y = x_{2n+1}$ in (2.1.1), we have

$$\|ABPx_{2n} - STx_{2n+1}\| \leq \alpha\left[\frac{\|P^2x_{2n} - ABPx_{2n}\|^3 + \|Qx_{2n+1} - STx_{2n+1}\|^3}{\|P^2x_{2n} - ABPx_{2n}\|^2 + \|Qx_{2n+1} - STx_{2n+1}\|^2}\right]$$
$$+\beta\left[\frac{\|P^2x_{2n} - ABPx_{2n}\|^2 + \|Qx_{2n+1} - STx_{2n+1}\|^2}{\|P^2x_{2n} - ABPx_{2n}\| + \|Qx_{2n+1} - STx_{2n+1}\|}\right]$$
$$+\gamma\|P^2x_{2n} - Qx_{2n+1}\|$$
$$+\delta[\|P^2x_{2n} - STx_{2n+1}\| + \|ABPx_{2n} - Qx_{2n+1}\|].$$
$$\leq \alpha[\|P^2x_{2n} - ABPx_{2n}\| + \|Qx_{2n+1} - STx_{2n+1}\|]$$
$$+\beta[\|P^2x_{2n} - ABPx_{2n}\| + \|Qx_{2n+1} - STx_{2n+1}\|]$$
$$+\gamma\|P^2x_{2n} - Qx_{2n+1}\|$$
$$+\delta[\|P^2x_{2n} - STx_{2n+1}\| + \|ABPx_{2n} - Qx_{2n+1}\|].$$

Letting $n\to\infty$ and using above results, we get

$$\|Pu - u\| \leq (\gamma + 2\delta)\|Pu - u\|.$$

So that
Pu = u.

Step 2. Putting x = u and $y = x_{2n+1}$ in (2.1.1), we have

$$\|ABu - STx_{2n+1}\| \leq \alpha\left[\frac{\|Pu - ABu\|^3 + \|Qx_{2n+1} - STx_{2n+1}\|^3}{\|Pu - ABu\|^2 + \|Qx_{2n+1} - STx_{2n+1}\|^2}\right]$$
$$+\beta\left[\frac{\|Pu - ABu\|^2 + \|Qx_{2n+1} - STx_{2n+1}\|^2}{\|Pu - ABu\| + \|Qx_{2n+1} - STx_{2n+1}\|}\right]$$
$$+\gamma\|Pu - Qx_{2n+1}\|$$
$$+\delta[\|Pu - STx_{2n+1}\| + \|ABu - Qx_{2n+1}\|].$$
$$\leq \alpha[\|Pu - ABu\| + \|Qx_{2n+1} - STx_{2n+1}\|]$$
$$+\beta[\|Pu - ABu\| + \|Qx_{2n+1} - STx_{2n+1}\|]$$
$$+\gamma\|Pu - Qx_{2n+1}\| + \delta[\|Pu - STx_{2n+1}\| + \|ABu - Qx_{2n+1}\|].$$

Letting $n\to\infty$ and using above results, we get

$$\|ABu - u\| \leq (\alpha + \beta + \delta)\|u - ABu\|.$$

So that
ABu = u.

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Step 3. Since AB(X) ⊂ Q(X), there exists a point v ∈ X such that ABu = u = Qv.
Putting x = u and y = v in (2.1.1), we get

$$\|ABu - STv\| \le \alpha\left[\frac{\|Pu-ABu\|^3+\|Qv-STv\|^3}{\|Pu-ABu\|^2+\|Qv-STv\|^2}\right]$$
$$+\beta\left[\frac{\|Pu-ABu\|^2+\|Qv-STv\|^2}{\|Pu-ABu\|+\|Qv-STv\|}\right]$$
$$+\gamma\|Pu-Qv\|+\delta[\|Pu-STv\|+\|ABu-Qv\|].$$
$$\le \alpha[\|Pu-ABu\|+\|Qv-STv\|]$$
$$+\beta[\|Pu-ABu\|+\|Qv-STv\|]$$
$$+\gamma\|Pu-Qv\|+\delta[\|Pu-STv\|+\|ABu-Qv\|].$$

Using above results, we get

$$\|u-STv\| \le (\alpha+\beta+\delta)\|u-STv\|.$$

So that
u = STv.
Now, STv = Qv, therefore v is the coincidence point of ST and Q. The weak compatibility of (Q, ST) implies that QSTv = STQv that is Qu = STu.

Step 4. Putting x = u and y = u in (2.1.1), we get

$$\|ABu - STu\| \le \alpha\left[\frac{\|Pu-ABu\|^3+\|Qu-STu\|^3}{\|Pu-ABu\|^2+\|Qu-STu\|^2}\right]$$
$$+\beta\left[\frac{\|Pu-ABu\|^2+\|Qu-STu\|^2}{\|Pu-ABu\|+\|Qu-STu\|}\right]$$
$$+\gamma\|Pu-Qu\|+\delta[\|Pu-STu\|+\|ABu-Qu\|].$$
$$\le \alpha[\|Pu-ABu\|+\|Qu-STu\|]$$
$$+\beta[\|Pu-ABu\|+\|Qu-STu\|]$$
$$+\gamma\|Pu-Qu\|+\delta[\|Pu-STu\|+\|ABu-Qu\|].$$

Using above results, we get

$$\|u-STu\| \le (\gamma+2\delta)\|u-STu\|.$$

So that
u = STu.
Hence z is a common fixed point of AB, ST, P and Q.

Step 5. Putting x = Au and y = u in (2.1.1), we get

$$\|AB(Au) - STu\| \le \alpha\left[\frac{\|P(Au)-AB(Au)\|^3+\|Qu-STu\|^3}{\|P(Au)-AB(Au)\|^2+\|Qu-STu\|^2}\right]$$
$$+\beta\left[\frac{\|P(Au)-AB(Au)\|^2+\|Qu-STu\|^2}{\|P(Au)-AB(Au)\|+\|Qu-STu\|}\right]$$
$$+\gamma\|P(Au)-Qu\|$$
$$+\delta[\|P(Au)-STu)\|+\|AB(Au)-Qu\|].$$
$$\le \alpha[\|P(Au)-AB(Au)\|+\|Qu-STu\|]$$
$$+\beta[\|P(Au)-AB(Au)\|+\|Qu-STu\|]$$
$$+\gamma\|P(Au)-Qu\|+\delta[\|P(Au)-STu\|+\|AB(Au)-Qu\|].$$

Since AB= BA implies ABu = BAu and AB (Au) = A (BAu)= Au.
Also PA = AP implies that PAu = APu = Au.

$$\|Au-u\| \le (\gamma+2\delta)\|Au-u\|.$$

So that

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Au = u.
Now
u = ABu = BAu = Bu.

Step 6. Putting x = u and y = Su in (2.1.1), we get

$$\|ABu - ST(Su)\| \le \alpha\left[\frac{\|Pu - ABu\|^3 + \|Q(Su) - ST(Su)\|^3}{\|Pu - ABu\|^2 + \|Q(Su) - ST(Su)\|^2}\right]$$
$$+\beta\left[\frac{\|Pu - ABu\|^2 + \|Q(Su) - ST(Su)\|^2}{\|Pu - ABu\| + \|Q(Su) - ST(Su)\|}\right]$$
$$+\gamma\|Pu - Q(Su)\|$$
$$+\delta[\|Pu - ST(Su)\| + \|ABu - Q(Su)\|].$$

Since ST = TS implies STu = TSu and ST (Su) = S(TSu) = Su.
Also QS = SQ implies that QSu = SQu = Su.
Using above results, we get

$$\|u - Su\| \le (\gamma + 2\delta)\|u - Su\|.$$

So that
u = Su.
Hence
u = STu = TSu = Tu.
Therefore, Au = Bu = Su = Tu = Pu = Qu = u.
Hence u is a common fixed point of A, B, S, T, P, and Q.

Case II. Now suppose AB is continuous and the pair (P, AB) is compatible, so we have
$AB^2x_{2n} \to ABu$ and $ABPx_{2n} \to ABu$ and $PABx_{2n} \to ABu$.

Step 7. Putting x= ABx_{2n} and y = x_{2n+1} in (2.1.1), we have

$$\|AB^2x_{2n} - STx_{2n+1}\| \le \alpha\left[\frac{\|PABx_{2n} - AB^2x_{2n}\|^3 + \|Qx_{2n+1} - STx_{2n+1}\|^3}{\|PABx_{2n} - AB^2x_{2n}\|^2 + \|Qx_{2n+1} - STx_{2n+1}\|^2}\right]$$
$$+\beta\left[\frac{\|PABx_{2n} - AB^2x_{2n}\|^2 + \|Qx_{2n+1} - STx_{2n+1}\|^2}{\|PABx_{2n} - AB^2x_{2n}\| + \|Qx_{2n+1} - STx_{2n+1}\|}\right]$$
$$+\gamma\|PABx_{2n} - Qx_{2n+1}\|$$
$$+\delta[\|PABx_{2n} - STx_{2n+1}\| + \|AB^2x_{2n} - Qx_{2n+1}\|].$$
$$\le \alpha[\|PABx_{2n} - AB^2x_{2n}\| + \|Qx_{2n+1} - STx_{2n+1}\|]$$
$$+\beta[\|PABx_{2n} - AB^2x_{2n}\| + \|Qx_{2n+1} - STx_{2n+1}\|]$$
$$+\gamma\|PABx_{2n} - Qx_{2n+1}\|$$
$$+\delta[\|PABx_{2n} - STx_{2n+1}\| + \|AB^2x_{2n} - Qx_{2n+1}\|].$$

Letting n→∞ and using above results, we get

$$\|ABu - u\| \le (\gamma + 2\delta)\|ABu - u\|.$$

So that
$ABu = u.$

Step 8. Since AB(X) ⊂ Q(X), there exists a point v ∈ X such that ABu = u = Qv.
Putting x = ABx_{2n} and y = v in (2.1.1), we get

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$$\|AB^2x_{2n} - STv\| \le \alpha\left[\frac{\|PABx_{2n} - AB^2x_{2n}\|^3 + \|Qv - STv\|^3}{\|PABx_{2n} - AB^2x_{2n}\|^2 + \|Qv - STv\|^2}\right]$$
$$+\beta\left[\frac{\|PABx_{2n} - AB^2x_{2n}\|^2 + \|Qv - STv\|^2}{\|PABx_{2n} - AB^2x_{2n}\| + \|Qv - STv\|}\right]$$
$$+\gamma\|PABx_{2n} - Qv\|$$
$$+\delta[\|PABx_{2n} - STv\| + \|AB^2x_{2n} - Qv\|].$$

Letting n→∞ and using above results, we get

$$\|u - STv\| \le \alpha[\|u - u\| + \|u - STv\|] + \beta[\|u - u\| + \|u - STv\|]$$
$$+\gamma\|u - u\| + \delta[\|u - STv\| + \|u - u\|].$$
$$\|u - STv\| \le (\alpha + \beta + \delta)\|u - STv\|.$$

So that

$u = STv = Qv.$

Therefore v is the coincidence point of ST and Q. The weak compatibility of (Q, ST) implies that QSTv = STQv that is Qu = STu.

Step 9. Putting x = x_{2n} and y = u in (2.1.1), we have

$$\|ABx_{2n} - STu\| \le \alpha\left[\frac{\|Px_{2n} - ABx_{2n}\|^3 + \|Qu - STu\|^3}{\|Px_{2n} - ABx_{2n}\|^2 + \|Qu - STu\|^2}\right]$$
$$+\beta\left[\frac{\|Px_{2n} - ABx_{2n}\|^2 + \|Qu - STu\|^2}{\|Px_{2n} - ABx_{2n}\| + \|Qu - STu\|}\right]$$
$$+\gamma\|Px_{2n} - Qu\| + \delta[\|Px_{2n} - STu\| + \|ABx_{2n} - Qu\|].$$

Letting n→∞ and using above results, we get

$$\|u - STu\| \le (\gamma + 2\delta)\|u - STu\|.$$

So that

$$u = STu.$$

implies that

$Qu = STu = ABu = u.$

Step 10. Since ST(X) ⊂ P(X), there exists a point w ∈ X such that u = STu = Pw, so that Putting x = w and y = u in (2.1.1), we get

$$\|ABw - STu\| \le \alpha\left[\frac{\|Pw - ABw\|^3 + \|Qu - STu\|^3}{\|Pw - ABw\|^2 + \|Qu - STu\|^2}\right]$$
$$+\beta\left[\frac{\|Pw - ABw\|^2 + \|Qu - STu\|^2}{\|Pw - ABw\| + \|Qu - STu\|}\right]$$
$$+\gamma\|Pw - Qu\| + \delta[\|Pw - STu\| + \|ABw - Qu\|].$$

Using above results, we get $\|ABw - u\| \le (\alpha + \beta + \delta)\|ABw - u\|$.

So that

$$ABw = u.$$

Now, $u = ABw = Pw$, therefore w is the coincidence point of P and AB. The weak compatibility of (P, AB) implies that PABw = ABPw that is Pu = ABu = u.

Hence u is a common fixed point of AB, ST, P and Q.

Now applying step 5 and step 6, we get u is a common fixed point of A, B, S, T, P and Q.

Uniqueness.

Let z be another common fixed point of A, B, S, T, P and Q, then

z = Az = Bz = Sz = Tz = Pz = Qz.
Putting x = u and y = z in (2.1.1), we get

$$\|u - z\| \leq (\gamma + 2\delta)\|u - z\|.$$

So that

$$u = z.$$

Therefore u is a unique common fixed point of A, B, S, T, P and Q.

Remark 1. In theorem 2.1, If we take (P, AB) and (Q, ST) both compatible pairs, then also theorem remains true, as compatibility implies weak compatibility.

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