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On the volume of n-balls

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ABSTRACT: In this paper, we give a survey on some recent results on the volume of n-ball in the Euclidean space $\mathbb{R}^n$.

KEYWORDS: Volume, n-balls, spheres.

I. INTRODUCTION

The n-balls or spheres in the Euclidean spaces is a basic object in mathematics. In calculus, geometry, topology,... the n-balls appear in many examples. In 2-dim spaces, we have the area πR^2, in 3-dim spaces, we have the volume $\frac{4}{3}\pi R^3$. But in higher dimension spaces, there is no way to draw the n-balls. Therefore, it is difficult to image them and compute their volume is not trivial problem.

How to compute their volume? And how small the n-ball when n tends to infinity? These are natural questions. It is well-known how to use the gamma function to compute the are or volume of an n-ball of the radius R. Many authors try to answer the above questions by different methods.

Firstly, we have the following definition of n-balls in the Euclidean $\mathbb{R}^n$.

Definition 1.1 The set

$$B_n(R) := \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1^2 + x_2^2 + \cdots + x_n^2 \leq R^2\},$$

where R is a positive number, is called a n-ball with radius R in the Euclidean space $\mathbb{R}^n, n \geq 1$.

1. Where $n = 1$, $B_1(R)$ is the interval $[-R; R]$.
2. Where $n = 2$, $B_2(R)$ is the circle with center $O(0; 0)$ and radius R:

$$B_2(R) := \{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 \leq R^2\}.$$

3. Where $n = 3$, $B_3(R)$ is the sphere (ball) with center $O(0; 0; 0)$ and radius R:

$$B_3(R) := \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 \leq R^2\}.$$

There is an improtant problem: Compute the volume of the n-ball $B_n(R)$.

There are many results on this problem, for instance, see [1,2,3,5,6]. Moreover, in [4], the authors compute the volume of n-simplex.

In this paper, we will study some methods in computing the volume of n-balls in the Euclidean spaces and we give a survey on the methods. These methods we refer in [1,3,4,5,6]. They are not new results.

The paper is organized as follows. Section II is preliminaries. Section III, we recall some methods in computation of the volume of n-balls in the Euclidean spaces.

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II. PRELIMINARIES

In this section, we recall the notions and some properties of Gamma and Beta functions.

- The Gamma function (Euler):

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1} dt, Re z > 0.$$

- The Beta function:

$$B(p,q) = \int_0^1 x^{p-1}\,(1-x)^{q-1} dx.$$

We have some properties of the Gamma and Beta functions:

1. $\Gamma(z+1) = z\Gamma(z), Re z > 0.$
2. $\Gamma(n+1) = n!$ với $n = 0,1,2, \dots.$
3. $\Gamma(\frac{1}{2}) = \sqrt{\pi}.$
4. $\Gamma(x+1) \sim \sqrt{2\pi x}(x/e)^x, (x \to \infty).$
5. $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$
6. $B(p,q) = B(q,p).$

Note that Property 4 is the Stirling formula in calculus.

III. THE VOLUME OF N-BALLS IN THE EUCLIDEAN SPACES

Theorem 3.1. *The volume of the n-ball, with the radius R, is the following formula:*

$$V_n(R) = \frac{\pi^{n/2} R^n}{\Gamma(\frac{n}{2}+1)}.$$

We give here 3 methods to prove Theorem 3.1.

The method 1 (see [1])

We take $\mathbb{R}^n = \mathbb{R}^{n-2} \times \mathbb{R}^2$. Then $(x_1, \dots, x_n) \in B_n(R)$ if and only if

$$x_1^2 + x_2^2 + \dots + x_{n-2}^2 + x_{n-1}^2 + x_n^2 \le R^2,$$

this is equivalent to

$$x_1^2 + x_2^2 + \dots + x_{n-2}^2 \le R^2 - x_{n-1}^2 - x_n^2.$$

Hence,

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$$\begin{aligned} V_n(R) &= \int_{B_n(R)} d\,x_1 dx_2 \dots dx_n \\ &= \int_{B_2(R)} \Big(\int_{B_{n-2}(\sqrt{R^2 - x_{n-1}^2 - x_n^2})} d\,x_1 \dots dx_{n-2} \Big) dx_{n-1} dx_n \end{aligned}$$

By the induction, we have:

$$V_n(R) = \frac{\pi^{(n-2)/2}}{\Gamma(\frac{n-2}{2}+1)} \int_{B_2(R)} (R^2 - x_{n-1}^2 - x_n^2)^{(n-2)/2} dx_{n-1} dx_n.$$

By using the polar coordinates, we have

$$\frac{\pi^{(n-2)/2}}{\Gamma(\frac{n}{2})} \int_0^{2\pi} d\,\theta \int_0^R (R^2 - t^2)^{(n-2)/2} t dt = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \cdot \frac{R^n}{n} = \frac{\pi^{n/2} R^n}{\Gamma(\frac{n}{2}+1)}.$$

The method 2 (see [5])

Since $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$, we have

$$\begin{aligned} V_n(R) &= \int_{B_n(R)} d\,x_1 dx_2 \dots dx_n \\ &= \int_{B_1(R)} \Big(\int_{B_{n-1}(\sqrt{R^2 - x_n^2})} d\,x_1 \dots dx_{n-1} \Big) dx_n, \end{aligned}$$

by the induction, we obtain

$$\begin{aligned} V_n(R) &= \frac{\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2}+1)} \int_{-R}^R (R^2 - x_n^2)^{(n-1)/2} dx_n \\ &= \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n+1}{2})} \int_0^R (R^2 - x_n^2)^{(n-1)/2} dx_n, \end{aligned}$$

put $x_n = R\sqrt{t}$, we have

$$\begin{aligned} V_n(R) &= \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n+1}{2})} \frac{R^n}{2} \int_0^1 (1-t)^{(n-1)/2} t^{-1/2} dt \\ &= R^n \frac{\pi^{(n-1)/2}}{\Gamma(\frac{n+1}{2})} B(\frac{n+1}{2}, \frac{1}{2}) \\ &= R^n \frac{\pi^{(n-1)/2}}{\Gamma(\frac{n+1}{2})} \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2}+1)} \end{aligned}$$

Since $\Gamma(\frac{1}{2}) = \pi^{1/2}$, we obtain $V_n(R) = \frac{\pi^{n/2} R^n}{\Gamma(\frac{n}{2}+1)}$.

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The method 3 (Lasserre's method)

Lasserre considered a functional and use the Laplace transform to prove the theorem (see [4]).

Let consider $f: \mathbb{R}^+ \to \mathbb{R}^+$,

$$y \mapsto f(y) := \int_{\|x\|^2 \le y} d\,x.$$

This function is the formula of the volume of sphere with radius $\sqrt{y}$. Let consider the Laplace transform $F: \mathbb{C} \to \mathbb{C}$ which is defined by:

$$z \mapsto F(z) := \int_0^\infty e^{-zy} f(y) dy, z \in \mathbb{C}, Re(z) > 0.$$

Then we have

$$\begin{aligned} F(z) &= \int_0^\infty e^{-zy} [\int_{\|x\|^2 \le y} d\,x] dy \\ &= \int_{\mathbb{R}^n} [\int_{\|x\|^2}^\infty e^{-zy}\, dy] dx \\ &= z^{-1} \int_{\mathbb{R}^n} e^{-z\|x\|^2}\, dx \\ &= z^{-1} \prod_{i=1}^n \int_{-\infty}^\infty e^{-zx_i^2}\, dx_i \\ &= z^{-1}[\pi/z]^{n/2} \\ &= z^{-n/2-1}\pi^{n/2} = \frac{\pi^{n/2}}{\Gamma(n/2+1)} \cdot \frac{\Gamma(n/2+1)}{z^{n/2+1}}. \end{aligned}$$

It is easy to see that $\frac{\Gamma(n/2+1)}{z^{n/2+1}}$ is an image of the Laplacian transform of $y^{n/2}$, i.e.

$$\frac{\Gamma(n/2+1)}{z^{n/2+1}} = \mathcal{L}(y^{n/2}).$$

Therefore,

$$f(y) = \frac{\pi^{n/2}}{\Gamma(n/2+1)} y^{n/2}.$$

By the properties of the Laplacian transform, we have:

$$\mathcal{L}(f) = \mathcal{L}(g) \Rightarrow f = g.$$

The theorem is proved.

We have the following consequence.

Corollary 3.2.

For $R > 0$, $\lim_{n\to\infty} V_n(R) = 0$.

It is easy to prove the corollary by using Stirling formula (Property 4).

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Moreover, some works give results on the volume of balls in the complex spaces $\mathbb{C}^n$. For instance, we recall Hijab's result (see [2]).

Theorem 3.3. (Hijab [2])

The volume of the unit balls

$$B = \{(z_1, \dots, z_n) \in \mathbb{C}^n: |z_1|^2 + \cdots + |z_n|^2 < 1\}$$

is $\pi^n/n!$.

Consequently, we have, $\lim_{n\to\infty} V_n(B) = 0$.

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