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On the global Holderian error bounds for differentiable semi-algebraic functions

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ABSTRACT: Based on the paper of Hà and Hoàng [1], we will extend some results of [1] on the global Holderian error bound (GHEB for short) for the sublevel set $[f \le t] := \{x \in \mathbb{R}^n \mid f(x) \le t\}$ from polynomial functions to differentiable semi-algebraic functions in n real variables.

Firstly, we review a formula in [1] for the set $H(f) = \{t \in \mathbb{R} \mid [f \le t] \text{ has a global Holderian error bound}\}$ via the so called special Fedoryuk values of f. Then, it is similar to [1], using this formula, we can determine all of stability types of a global Holderian error bound for $[f \le t]$, where $t \in \mathbb{R}$. Moreover, we point out some different points of the case of semi-algebraic functions by some examples.

KEYWORDS: Global Holderian error bounds, semi-algebraic functions, stability, Fedoryuk values, semi-algebraic optimization.

I. INTRODUCTION

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a semi-algebraic functions. For $t \in \mathbb{R}$, put $[f \le t] := \{x \in \mathbb{R}^n \mid f(x) \le t\}$ and $[a]_+ = \max\{a, 0\}$.

Definition 1.1 ([2]) We say that the non-empty set $[f \le t]$ has a *global Holderian error bound* (GHEB for short) if there exist $\alpha, \beta, c > 0$ such that

$$[f(x) - t]_+^{\alpha} + [f(x) - t]_+^{\beta} \ge c\,\mathrm{dist}(x, [f \le t]), \text{ for all } x \in \mathbb{R}^n.$$

The existence of error bounds has many important applications, including convergence analysis in optimization problems, variational inequalities, identifying active constraints, sensitive analysis…(see, for example [2, 6, 8, 11, 12]). The study of error bounds has received rising attention in many papers of mathematical programming in recent years (see [2, 6, 8, 11,12]).

In this paper, we will study stability of global Holderian error bound for the set $[f \le t]$ under a perturbation of t in the case of differentiable semi-algebraic functions. The following natural questions arise:

1. Suppose that $[f \le t]$ has a GHEB, when does there exist an open interval $I(t)$, $t \in I(t)$ such that for any $t' \in I(t)$, $[f \le t']$ has also a GHEB?
2. Assume that $[f \le t]$ does not have a GHEB, when does there exist an open interval $I(t)$, $t \in I(t)$ such that for any $t' \in I(t)$, $[f \le t']$ does not also have a GHEB?

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Let us consider $H(f)=\{t\in\mathbb{R} \mid [f\le t] \text{ has a global Holderian error bound}\}$

It is similar to [1], we can show that the relationship between the set $H(f)$ and the so-called the Fedoryuk set, i.e.

$$\tilde{K}_{\infty}(f):=\left\{\begin{matrix} t\in\mathbb{R}:\exists\{x^k\}\subset\mathbb{R}^n, \|x^k\|\to\infty, \\ \|\nabla f(x^k)\|\to 0, f(x^k)\to t \end{matrix}\right\}.$$

The Fedoryuk set is considered as generalized critical values and it relates to topology of algebraic and semi-algebraic sets (see [2, 6, 9]).

In this paper, we point out that in the case of differentiable semi-algebraic functions, we can obtain some similar results in [1]. However, in the case of two variables, there are some different things. Explicitly, the Fedoryuk set of a polynomial in two variables is always finite while the Fedoryuk set of a semi-algebraic function can be infinite.

The paper is organized as follows. In Section II, we give a short introduction to semi-algebraic geometry. Section III contains the formula for $H(f)$. We will give some properties of the Fedoryuk values, $H(f)$ and establish their relations in Section IV. Besides, we give the result on stability of Holderian error bounds in this Section. We will compare the case of polynomial and semi-algebraic functions in Section V.

II. PRELIMINARIES

In this section, we will recall the basic notions and some results of semi-algebraic geometry which can be found [3-6].

Definition 2.1 A semi-algebraic subset of $\mathbb{R}^n$ is the subset of $(x_1,\dots,x_n)$ in $\mathbb{R}^n$ satisfying a Boolean combination of polynomial equations and inequalities with real coefficients. In other words, the semi-algebraic subsets of $\mathbb{R}^n$ form the smallest class SA_n of subsets of $\mathbb{R}^n$ such that

1) If $P\in\mathbb{R}[x_1,\dots,x_n]$,then $\{x\in\mathbb{R}^n \mid P(x)=0\}\in SA_n$ and $\{x\in\mathbb{R}^n \mid P(x)>0\}\in SA_n$;

2) If $A\in SA_n$ and $B\in SA_n$, then $A\cup B, A\cap B$ and $\mathbb{R}^n\setminus A$ are in SA_n.

The following proposition gives us a structure of a semi-algebraic set

Proposition 2.2 ([3-6]) *Every semi-algebraic subset of $\mathbb{R}^n$ is the union of finitely many semi-algebraic subsets of the form*

$$\{x\in\mathbb{R}^n \mid P(x)=0, Q_1(x)>0,\dots,Q_l(x)>0\}, \text{ where } l\in\mathbb{N} \text{ and } P,Q_1,\dots,Q_l\in\mathbb{R}[x_1,\dots,x_n].$$

The following theorem plays the important role to proving many results in real algebraic geometry

Theorem 2.3 ([3-6]) *Let A be a semi-algebraic subset of $\mathbb{R}^{n+1}$ and $\pi:\mathbb{R}^{n+1}\to\mathbb{R}^n$ be a projection on the first n coordinates. Then $\pi(A)$ is a semi-algebraic subset of $\mathbb{R}^n$.*

Definition 2.4 ([3-6]) Let $A\subset\mathbb{R}^n$. A function $f:A\to\mathbb{R}$ is said to be *semi-algebraic* if its graph

$\{(x,y)\in\mathbb{R}^n\times\mathbb{R}: y=f(x)\}$ is a semi-algebraic subset in $\mathbb{R}^n\times\mathbb{R}$.

We list basic properties of semi-algebraic sets and functions.

1. The class of semi-algebraic sets is closed under Boolean operators, taking Cartesian product, closure and interior.

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2. A composition of semi-algebraic maps is a semi-algebraic map; the image and pre-image of a semi-algebraic set under a semi-algebraic map are semi-algebraic sets.

3. If S is a semi-algebraic set, then the distance function $\begin{array}{l}\operatorname{dist}(.,S):\mathbb{R}^n \to \mathbb{R},\\ x \mapsto \operatorname{dist}(x,S) := \inf\left\{\|x-y\| : y \in S\right\}\end{array}$ is also semi-algebraic.

III. THE EXISTENCE OF GLOBAL HOLDERIAN ERROR BOUNDS FOR DIFFERENTIABLE SEMI-ALGEBRAIC FUNCTIONS

Let $f:\mathbb{R}^n \to \mathbb{R}$ be a semi-algebraic function and $t \in [\inf f, +\infty)$.

Definition 3.1 ([2, 6]) Let S be a subset of $\mathbb{R}^n$. We say that

i) A sequence $\left\{x^k\right\} \subset \mathbb{R}^n$ is said to be of the first type of $[f \le t]$ w.r.t S if

$$\begin{array}{l}\left\|x^k\right\| \to \infty,\\ f(x^k) > t, f(x^k) \to t,\\ \exists \delta > 0 : \operatorname{dist}\left(x^k, [f \le t]\right) \ge \delta > 0,\\ \text{and } \left\{x^k\right\} \subset S.\end{array}$$

ii) A sequence $\left\{x^k\right\} \subset \mathbb{R}^n$ is said to be of the first type of $[f \le t]$ w.r.t S if

$$\begin{array}{l}\left\|x^k\right\| \to \infty,\\ \exists M \in \mathbb{R}^n : t < f(x^k) \le M < +\infty,\\ \operatorname{dist}\left(x^k, [f \le t]\right) \to +\infty,\\ \text{and } \left\{x^k\right\} \subset S.\end{array}$$

The following theorem gives us a necessary and sufficient condition for the existence of a global Holderian error bound for a semi-algebraic function, it was proved in [7] and it is an extended result of Theorem A in [2]:

Theorem 3.2 *Let $f:\mathbb{R}^n \to \mathbb{R}$ be a semi-algebraic function. Then the following statements are equivalent:*

i) *There are no sequences of the first or second types of $[f \le t]$ w.r.t. $\mathbb{R}^n$;*

ii) *$[f \le t]$ has a global Holderian error bound, i.e. there exist $\alpha, \beta, c > 0$ such that*

$$[f(x)-t]_+^{\alpha} + [f(x)-t]_+^{\beta} \ge c\operatorname{dist}\left(x, [f \le t]\right),$$

for all $x \in \mathbb{R}^n$.

Remark 3.3 We can extend this theorem to definable function in an o-minimal structure (see [8]), an object in real algebraic geometry. The o-minimal structure is the structure which is larger than structure of semi-algebraic.

We denote

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$$F^1 = \begin{Bmatrix} t\in\mathbb{R}:[f\le t] \text{ has a sequence} \\ \text{of the first type w.r.t } \mathbb{R}^n \end{Bmatrix} \text{ and } F^2 = \begin{Bmatrix} t\in\mathbb{R}:[f\le t] \text{ has a sequence} \\ \text{of the second type w.r.t } \mathbb{R}^n \end{Bmatrix}.$$

We have a useful property of the set F^2:

Lemma 3.3 *Assume that the set* F^2 *is not an empty-set. Then* $t\in F^2$ *implies that* $t'\in F^2, \forall t'\in[\inf f, t]$.

Proof. By the assumption, there exists a sequence of the second type via t. Suppose that it is $\{x^k\}$, then $\|x^k\|\to\infty, \exists M\in\mathbb{R}: t<f(x^k)\le M$ such that $\text{dist}\left(x^k,[f\le t]\right)\to+\infty$. Since $\inf f\le t'\le t$, we have $[f\le t']\subseteq[f\le t]$.

Hence, $\text{dist}\left(x^k,[f\le t']\right)\ge\text{dist}\left(x^k,[f\le t]\right)$. In other hand, we have $\text{dist}\left(x^k,[f\le t']\right)\to+\infty$. This implies that $\{x^k\}$ is a sequence of the second type of $[f\le t']$, or $t'\in F^2$.

Definition 3.4 ([1]) Put $h(f)=\begin{cases}\sup\{t\in\mathbb{R}:t\in F^2\} & \text{if } F^2\ne\varnothing,\\ \inf f & \text{if } F^2=\varnothing.\end{cases}$

We call $h(f)$ the *threshold* of global Holderian error bounds of f.

Remark 3.5 By Lemma 3.3, we have the threshold $h(f)$ is well-defined.

We extend the following theorem from the polynomial functions (see [1]) to differentiable semi-algebraic functions:

Theorem 3.6 (The formula for $H(f)$) *We have*

$$H(f)=\begin{cases}(h(f),+\infty)\setminus F^1 & \text{if } h(f)\in F^2,\\ [h(f),+\infty)\setminus F^1 & \text{if } h(f)\notin F^2,\\ [\inf f,+\infty)\setminus F^1 & \text{if } F^2=\varnothing \text{ and } \inf f>-\infty,\\ \mathbb{R}\setminus F^1 & \text{if } F^2=\varnothing \text{ and } \inf f=-\infty.\end{cases}$$

Proof. From Theorem 3.2, we have $[f\le t]$ has a global Holderian error bound if and only if $t\notin F^1\cup F^2$ and by Lemma 3.3, similar to [1, Theorem 3.2], we have the proof of this theorem.

IV. FEDORYUK VALUES, THE EXISTENCE AND STABILITY OF GLOBAL HOLDERIAN ERROR BOUNDS

We will establish some similar results to [1]. Moreover, we point out some different facts of polynomial functions case and semi-algebraic functions case.

a. The existence of global Holderian error bounds

Definition 4.1 Let $f:\mathbb{R}^n\to\mathbb{R}$ be a differential semi-algebraic function. The set of *Fedoryuk values of* f *(or Fedoryuk set)* is defined by

$$\tilde{K}_\infty(f):=\begin{Bmatrix} t\in\mathbb{R}:\exists\{x^k\}\subset\mathbb{R}^n, \|x^k\|\to\infty, \\ \|\nabla f(x^k)\|\to 0, f(x^k)\to t \end{Bmatrix}.$$

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Remark 4.2. In the paper [9], the authors point out that the Fedoryuk set can be infinite. For example, consider the polynomial:

$$f(x,y,z)=x+x^2y+x^4yz.$$

We have $\tilde{K}_\infty(f)=\mathbb{R}$. Moreover,

$$\tilde{K}_\infty(f^2)=[0,+\infty).$$

The following propositions give us a property of the sets F^1 and F^2. The proofs are similar to Proposition 4.1 in [1] (see also [2]) by using the Ekeland Variational Principle (see, for example, [12]).

Proposition 4.3 $F^1\subset\tilde{K}_\infty(f)$.

Proposition 4.4 *If the set* $[f\le t]$ *has a sequence* $\{x^k\}_k$ *of the second type, then there is a constant* $M>t$ *and a sequence* $\{y^k\}_k$ *of the second type for* $[f\le t]$ *such that* $\|\nabla f(y^k)\|\to 0$ *and* $\lim_{k\to\infty} f(y^k)\in\tilde{K}_\infty(f)$. *Moreover,* $[t,M]\cap\tilde{K}_\infty(f)\neq\varnothing$.

We give some properties of the set $H(f)$ and the threshold $h(f)$:

1. *If* $h(f)\neq\pm\infty$, *then* $h(f)\in\tilde{K}_\infty(f)$.
2. *If* $\#\tilde{K}(f)<+\infty$, *then* $H(f)\neq\varnothing$.
3. *If* $F^2(f)=\varnothing$, *then* $h(f)=-\infty$.

The set $H(f)$ *is a semi-algebraic subset of* $\mathbb{R}$. *Hence,* $H(f)$ *is either empty or a finite disjoint union of points and intervals.*

b. Stability of global Holderian error bounds under perturbations of parameter

From the structure of $H(f)$, we have the definition of all types of stability of error bounds.

Definition 4.5 ([1]) Let $t\in[\inf f,+\infty)$.

1. The value t is said to be *y-stable* if $t\in H(f)$ and there exists an open interval $I(t)$ such that $t\in I(t)\subset H(f)$;
2. The value t is said to be *y-right stable* if $t\in H(f)$ and there exists $\varepsilon>0$ such that $[t,t+\varepsilon)\subset H(f)$ and $(t-\varepsilon,t)\cap H(f)=\varnothing$;
3. The value t is said to be *y-left stable* if $t\in H(f)$ and there exists $\varepsilon>0$ such that $(t-\varepsilon,t]\subset H(f)$ and $(t,t+\varepsilon)\cap H(f)=\varnothing$;
4. The value t is said to be *y-isolated* if $t\in H(f)$ and for all $\varepsilon>0$ sufficiently small, $[(t-\varepsilon,t)\cup(t,t+\varepsilon)]\subset\mathbb{R}\setminus H(f)$;
5. The value t is called *n-stable* if $t\in[\inf f,+\infty)\setminus H(f)$ and there exists an open interval $I(t)$ such that $t\in I(t)\subset[\inf f,+\infty)\setminus H(f)$;
6. The value t is called *n-right stable* if $t\in[\inf f,+\infty)\setminus H(f)$ and there exists $\varepsilon>0$ such that $[t,t+\varepsilon)\subset[\inf f,+\infty)\setminus H(f)$ and $(t-\varepsilon,t)\cap([\inf f,+\infty)\setminus H(f))=\varnothing$;
7. The value t is called *n-left stable* if $t\in[\inf f,+\infty)\setminus H(f)$ and there exists $\varepsilon>0$ such that $(t-\varepsilon,t]\subset[\inf f,+\infty)\setminus H(f)$ and $(t,t+\varepsilon)\cap([\inf f,+\infty)\setminus H(f))=\varnothing$;

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8. The value t is called *n-isolated* if $t \in [\inf f, +\infty) \setminus H(f)$ and for all $\varepsilon > 0$ sufficiently small, $(t-\varepsilon, t) \cup (t, t+\varepsilon) \subset ([\inf f, +\infty) \setminus H(f))$;

We can establish similar results in [1] on stability of GHEB in the case of semi-algebraic functions. If the Fedoryuk set is finite, then the following theorem is an extension of a result in [1] from the polynomials case to the differentiable semi-algebraic functions:

Theorem 4.6 *Let $\tilde{K}_\infty(f)$ be a non-empty set and $t \in [\inf f, +\infty)$. Then, t is one of the following types*

Case A: If $h(f) = -\infty$, then

- *i.* *t is y-stable if and only if $t \notin F^1$;*
- *ii.* *t is n-isolated point if and only if $t \in F^1$;*

Case B: If $h(f)$ is a finite value, then

1. *t is y-stable if and only if $t > h(f)$ and $t \notin F^1$;*
2. *t is y-right stable if and only if $t = h(f) \in H(f)$;*

- *a.* *t is n-stable if and only if $\inf f < t < h(f)$;*
- *b.* *t is n-right stable if and only if $t = \inf f < h(f)$ and $f^{-1}(\inf f) \neq \emptyset$;*
- *c.* *t is n-left stable if and only if $t = h(f) \notin H(f)$;*
- *d.* *t is a n-isolated point if and only if $t > h(f)$ and $t \in F^1$.*

Note that, if we have type 2, then we have no type b and vice versa.

V. COMPARING POLYNOMIAL AND SEMI-ALGEBRAIC FUNCTIONS

We focus to the case of two variables. The authors in [1] gave a proof which says that the set of Fedoryuk values is finite, i.e., $\#\tilde{K}_\infty(f) < +\infty$, where f is a polynomial function. We give a following proposition to prove difference between two kinds of functions:

Proposition 5.1 *There exist some differentiable semi-algebraic functions such that their Fedoryuk set is infinite in the case of two variables.*

Proof. We only need to point out an example satisfying the proposition. Let us consider the following semi-algebraic functions

$$f(x, y) = \frac{y}{1+x^2}.$$

If we take sequence $\{x^k\}_k$ such that $x^k = (k, a(1+k^2)), a \in \mathbb{R},$ then we have $\|x^k\| \to \infty, f(x^k) = a, \nabla f(x^k) \to 0.$ This implies that $\tilde{K}_\infty(f) = \mathbb{R}$.

Let us consider the function $g = \dfrac{y^2}{1+x^2}$. If we take sequence $\{x^k\}_{k=1}^{\infty}$ such that $x^k = \left(k, \sqrt{a(1+k^2)}\right), a \in \mathbb{R}, a \geq 0,$ then we have $\|x^k\| \to \infty, f(x^k) = a, \nabla f(x^k) \to 0.$ This implies that $\tilde{K}_\infty(g) = [0, +\infty)$.

Remark 5.2 We can consider the functions f, g in above proposition as n variables. For example, we can take

$$f(x_1, x_2, ..., x_n) = \frac{x_2}{1+x_1^2}, g(x_1, x_2, ..., x_n) = \frac{x_2^2}{1+x_1^2}.$$

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