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Numerical Solution of the Navier-Stokes Equations in the "Vortex-Current" System

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ABSTRACT: In the numerical solution of the Navier-Stokes system of equations, this system is represented as a system of equations for the current function and vortex. Moreover, the resulting system belongs to two types: an elliptic equation for the current function and a nonlinear parabolic equation for a vortex.

The main purpose of this article is to develop an efficient and high-precision numerical algorithm for joint solution of the current function and vortex equation, as well as to construct and study the behavior of the current function level line and vortex in a wide range of time variation.

The upper relaxation iteration scheme is compared with the simple iteration scheme in terms of the number of iterations to obtain the required accuracy. Show the efficiency of the applied method and the rather fast convergence of the iterative solution to the exact one in comparison with the simple iteration method.

For the numerical solution of the current function equation, a number of methods, both direct and iterative, have been developed. However, the question of the effectiveness of these methods remains relevant. In this paper, the equations for the current function are solved using an iterative upper relaxation scheme. The performed numerical calculations show the effectiveness of the applied method.

The nonlinear parabolic equation for a vortex is approximated using two implicit difference schemes: both according to the Peasman-Reckford scheme and according to the Douglas-Reckford scheme. The results of calculations carried out according to these schemes illustrate that the Peasman-Rekford scheme has high accuracy. For numerical simulation of the Navier-Stokes equations in the vortex-current system, alternating direction methods (Peasman-Reckford and Douglas-Reckford) and the iterative upper relaxation method are used. A large-scale computational experiment was carried out using the trial function method. Comparison of the selected trial function with the obtained difference solutions at the grid nodes, the level lines of the current function and vortex are plotted at different times. Based on these results, graphs are plotted and numerical results are presented. The obtained numerical results show the efficiency and high accuracy of the Peasman-Reckford difference scheme when used together with the iterative upper relaxation scheme for numerical simulation of the Navier-Stokes equations in the “vortex-current” system.

KEYWORDS: the Navier-Stokes equations, "vortex-current" system, the alternating direction method, scheme of Peasman-Reckford, scheme of Douglas-Reckford, the iterative upper relaxation scheme

1. INTRODUCTION

To date, there are numerous studies devoted to the numerical simulation of viscous incompressible fluid problems based on two-dimensional Navier-Stokes equations in the "vortex-current" system.

Despite this, the issue of the effectiveness of the application of certain methods for numerical modeling of the above problem is urgent.

In [1], a method was proposed for solving the Navier-Stokes equations in natural variables. The method is based on the joint solution of the equation of motion and the equation of continuity with the use of finite difference approximation in [2], a numerical method for solving the Navier-Stokes equations of a viscous incompressible fluid (in physical

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variables) is proposed, supplemented by the equations of heat conduction. When constructing it, an approximate factorization scheme is used with the splitting of the original operators into physical processes in a special way. In [3], a new approach to calculating pressure was proposed when solving the complete Navier-Stokes equations in the "velocity-pressure" variables on structured grids. The method is based on the use of integral forms of the equation of continuity and pressure decomposition, on the basis of which an auxiliary problem is formulated.

In [4], the complete system of Navier-Stokes equations in the velocity-pressure variables is solved by the numerical finite difference method for the case of a viscous incompressible fluid. The discretization of the original equations is implemented in spaced grids. In [5], algorithms for the numerical solution of the Navier-Stokes equations using high-performance computing technology, such as multiprocessor systems with distributed memory and graphics accelerators, are presented. In [6], the efficiency of an implicit iterative nonlinear recurrent method for solving systems of difference elliptic equations, arising in the numerical simulation of two-dimensional flows of a viscous incompressible fluid, is analyzed.

The work [7] is devoted to the study of numerical methods for solving the Navier-Stokes equations in variables "stream function-vortex". To solve the corresponding linear grid equations, a standard library was used, which contains an efficient parallel LU decomposition of matrices for solving systems of linear equations with expanded matrices.

In contrast to these works, in this work for the numerical solution of the "vortex-current" system of equations, methods of alternating directions (the Pismen-Reckford and Douglas-Reckford schemes) and the iterative method of upper relaxation are used together. In this case, the vortex equations (1) are approximated with two schemes, both Peaseman-Reckford and Douglas-Reckford, and equation (2) for the stream function is solved using the iterative upper relaxation method. The numerical results obtained by the Peasman-Reckford and Douglas-Rekford schemes are compared. The efficiency and high accuracy of the application of the Peasman-Reckford scheme with a combination of the iterative upper relaxation scheme for the numerical simulation of the Navier-Stokes equations in the "eddy-current" system is illustrated.

II. MATERIALS AND METHODS

A. Formulation of the problem.

In Cartesian coordinates, the system of Navier-Stokes equations in the form of "vortex-current" is written as follows [8].

$$\frac{\partial \omega}{\partial t}+\frac{\partial \omega}{\partial x}\frac{\partial \psi}{\partial y}-\frac{\partial \omega}{\partial y}\frac{\partial \psi}{\partial x}=\nu\left(\frac{\partial^2 \omega}{\partial x^2}+\frac{\partial^2 \omega}{\partial y^2}\right)+Q(t,x,y), \tag{1}$$

$$\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}=-\omega, \tag{2}$$

$$\frac{\partial \psi}{\partial y}=u,\quad \frac{\partial \psi}{\partial x}=-v,\quad \omega=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}. \tag{3}$$

Here, x, y - spatial coordinates, t - time, u and v - projection of the velocity vector on the coordinate axis, ν - coefficient of kinematic viscosity, ψ - stream function, ω - vortex function, Q - known function.

For system (1) - (2) in the domain $\overline{D}:\{(x,y,t)\in[0,1]\times[0,1]\times[0,T]\}$, we set the following boundary conditions:

$$\psi|_{x=0}=0,\quad \psi|_{x=1}=0,\quad \left.\frac{\partial \psi}{\partial x}\right|_{x=0}=0,\quad \left.\frac{\partial \psi}{\partial x}\right|_{x=1}=0,\quad 0\le y\le 1, \tag{4}$$

$$\psi\big|_{y=0}=0,\quad \psi\big|_{y=1}=0,\quad \left.\frac{\partial \psi}{\partial y}\right|_{x=0}=0,\quad \left.\frac{\partial \psi}{\partial y}\right|_{x=1}=0,\quad 0\le x\le 1, \tag{5}$$

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the initial conditions at $t=0$ are as follows:

$$\psi(0,x,y)=0, \quad \omega(0,x,y)=0. \qquad (6)$$

B. Finite difference approximation.

In the domain $\overline{D}$, we introduce a uniform grid in spatial coordinates x, y

$\overline{\Omega}_h=\left\{x_i=ih,\ y_j=jh,\ 0\le i,j\le N,\ h=\frac{1}{N}\right\}$ and a grid by time t $\overline{\Omega}_\tau=\{t_k=k\tau,\ k=0,1,...M\}$, where $\tau=T/M$.

The system of differential equations (1), (2) is approximated on a difference grid $\overline{\Omega}=\overline{\Omega}_h\times\overline{\Omega}_\tau$.

For the numerical solution of the vortex equation (1), we apply the method of alternating directions (the Peasman-Reckford scheme and the Douglas-Reckford scheme) [9]. It is known that both schemes, since implicit schemes are absolutely stable. The Peasman-Reckford scheme has an approximation error $\theta(\tau^2+|h|^2)$, and for the Douglas-Reckford scheme the corresponding error $\theta(\tau+|h|^2)$, where $|h|^2=h_1^2+h_2^2$, h_1, h_2 are grid steps in the variables x and y, respectively.

At first, equations (1) are approximated by the longitudinal-transverse difference scheme or the Peasman-Reckford scheme. In this scheme, the transition from layer k to layer $k+1$ is carried out in two stages. At the first stage, intermediate values of $\omega_{i,j}^{k+1/2}$ are determined from the following system of equations

$$\frac{\omega_{i,j}^{k+1/2}-\omega_{i,j}^{k}}{0,5\tau}+\frac{\omega_{i+1,j}^{k+1/2}-\omega_{i-1,j}^{k+1/2}}{2h}\frac{\psi_{i,j+1}^{k}-\psi_{i,j-1}^{k}}{2h}-\frac{\omega_{i,j+1}^{k}-\omega_{i,j-1}^{k}}{2h}\frac{\psi_{i+1,j}^{k}-\psi_{i-1,j}^{k}}{2h}=$$

$$=\frac{\nu}{h^2}\left(\omega_{i+1,j}^{k+1/2}-2\omega_{i,j}^{k+1/2}+\omega_{i-1,j}^{k+1/2}\right)+\frac{\nu}{h^2}\left(\omega_{i,j+1}^{k}-2\omega_{i,j}^{k}+\omega_{i,j-1}^{k}\right)+Q\left(t_{k+1/2},x_i,y_i\right), \qquad (7)$$

$$i,j=1,2,...,N-1 \quad k=0,1,...,M-1,$$

and at the second stage, using the found values of $\omega_{i,j}^{k+1/2}$, the values of $\omega_{i,j}^{k+1}$ are determined from the system of equations

$$\frac{\omega_{i,j}^{k+1}-\omega_{i,j}^{k+1/2}}{0,5\tau}+\frac{\omega_{i+1,j}^{k+1/2}-\omega_{i-1,j}^{k+1/2}}{2h}\frac{\psi_{i,j+1}^{k}-\psi_{i,j-1}^{k}}{2h}-\frac{\omega_{i,j+1}^{k+1}-\omega_{i,j-1}^{k+1}}{2h}\frac{\psi_{i+1,j}^{k}-\psi_{i-1,j}^{k}}{2h}=$$

$$=\frac{\nu}{h^2}\left(\omega_{i+1,j}^{k+1/2}-2\omega_{i,j}^{k+1/2}+\omega_{i-1,j}^{k+1/2}\right)+\frac{\nu}{h^2}\left(\omega_{i,j+1}^{k+1}-2\omega_{i,j}^{k+1}+\omega_{i,j-1}^{k+1}\right)+Q(t_{k+1},x_i,y_i), \qquad (8)$$

$$i,j=1,2,...,N-1 \quad k=0,1,...,M-1.$$

As noted above, the Douglas-Rackford scheme can be used to solve the vortex equation (1). The system of difference equations obtained according to this scheme have the form:

$$\frac{\omega_{i,j}^{k+1/2}-\omega_{i,j}^{k}}{\tau}+\frac{\omega_{i+1,j}^{k+1/2}-\omega_{i-1,j}^{k+1/2}}{2h}\frac{\psi_{i,j+1}^{k}-\psi_{i,j-1}^{k}}{2h}-\frac{\omega_{i,j+1}^{k}-\omega_{i,j-1}^{k}}{2h}\frac{\psi_{i+1,j}^{k}-\psi_{i-1,j}^{k}}{2h}=$$

$$=\frac{\nu}{h^2}\left(\omega_{i+1,j}^{k+1/2}-2\omega_{i,j}^{k+1/2}+\omega_{i-1,j}^{k+1/2}\right)+\frac{\nu}{h^2}\left(\omega_{i,j+1}^{k}-2\omega_{i,j}^{k}+\omega_{i,j-1}^{k}\right)+0,5Q\left(t_k,x_i,y_i\right), \qquad (9)$$

$$i,j=1,2,...,N-1 \quad k=0,1,...,M-1,$$

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$$\frac{\omega_{i,j}^{k+1}-\omega_{i,j}^{k+1/2}}{\tau}-\frac{\omega_{i,j+1}^{k+1}-\omega_{i,j-1}^{k+1}}{2h}\frac{\psi_{i+1,j}^{k}-\psi_{i-1,j}^{k}}{2h}=\frac{\nu}{h^2}\left(\omega_{i,j+1}^{k+1}-2\omega_{i,j}^{k+1}+\omega_{i,j-1}^{k+1}\right)-$$

$$-\frac{\nu}{h^2}\left(\omega_{i,j+1}^{k}-2\omega_{i,j}^{k}+\omega_{i,j-1}^{k}\right)+0,5Q(t_{k+1},x_i,y_i), \quad (10)$$

$i,j=1,2,...,N-1\ \ k=0,1,...,M-1.$

The algorithm for solving the system of difference equations to (9), (10) is similar to the above algorithms of the Peasman-Reckford scheme.

Equation (2) for the stream function is approximated by the following difference scheme

$$\frac{\psi_{i+1,j}^{k+1}-2\psi_{i,j}^{k+1}+\psi_{i-1,j}^{k+1}}{h^2}+\frac{\psi_{i,j+1}^{k+1}-2\psi_{i,j}^{k+1}+\psi_{i,j-1}^{k+1}}{h^2}=-\omega_{ij}^{k+1} \text{ or}$$

$$\frac{\psi_{i+1,j}^{k+1}+\psi_{i-1,j}^{k+1}+\psi_{i,j+1}^{k+1}+\psi_{i,j-1}^{k+1}-4\psi_{i,j}^{k+1}}{h^2}=-\omega_{ij}^{k+1} \quad (11)$$

$i,j=1,2,...,N-1\ \ k=0,1,...,M-1.$

Equation (3) is approximated as follows

$$u_{i,j}^{k+1}=\frac{\psi_{i,j+1}^{k+1}-\psi_{i,j-1}^{k+1}}{2h},\quad v_{i,j}^{k+1}=-\frac{\psi_{i+1,j}^{k+1}-\psi_{i-1,j}^{k+1}}{2h} \quad (12)$$

$i,j=1,2,...,N-1\ \ k=0,1,...,M-1.$

III. RESULT AND DISCUSSION.

Let us rewrite the equation (11) in the following convenient form for applying the method of simple iteration [10]

$$\psi_{ij}^{k+1,s+1}=\psi_{ij}^{k+1,s}+r_{ij}, \quad (13)$$

where

$$r_{ij}=\frac{\psi_{i+1,j}^{k+1,s}+\psi_{i-1,j}^{k+1,s}+\psi_{i,j+1}^{k+1,s}+\psi_{i,j-1}^{k+1,s}-4\psi_{i,j}^{k+1,s}+h^2w_{ij}^{k+1}}{4}. \quad (14)$$

The convergence rate of iteration is improved by using the sequential upper relaxation method [11]. The sequential over relaxation method uses the following iterative scheme

$$\psi_{ij}^{k+1,s+1}=\psi_{ij}^{k+1,s}+\vartheta\left(\frac{\psi_{i+1,j}^{k+1,s}+\psi_{i-1,j}^{k+1,s}+\psi_{i,j+1}^{k+1,s}+\psi_{i,j-1}^{k+1,s}-4\psi_{i,j}^{k+1,s}+h^2w_{ij}^{k+1}}{4}\right)=\psi_{ij}^{k+1,s}+\vartheta r_{ij}, \quad (15)$$

where the parameter ϑ belongs to the region $1\le\vartheta<2$. The optimal choice of the parameter ϑ depends on the eigenvalues of the iterative matrices for linear systems and for the problem under consideration is given by the following formula [11].

$$\vartheta=\frac{4}{2+\sqrt{4-4\cos^2\frac{\pi}{N-1}}}=\frac{4}{2+2\sqrt{1-\cos^2\frac{\pi}{N-1}}}=\frac{2}{1+\sin\frac{\pi}{N-1}}.$$

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The solution of the difference equations for the stream function (11) by the simple iteration method, the number of iterations $n_0(\varepsilon)$ required to achieve a given accuracy ε is determined by the formula [12].

$$n_0(\varepsilon) \approx \frac{2\ln(1/\varepsilon)}{\pi^2 h^2}, \qquad (16)$$

the corresponding number of iterations for the upper relaxation method is:

$$n_0(\varepsilon) \approx \frac{2\ln(1/\varepsilon)}{\pi h}. \qquad (17)$$

If $\varepsilon = 10^{-3}$, $h = 0.05$, the number of iterations $n_0(\varepsilon)$ calculated by formulas (16), (17) are approximately equal to 560 and 88. It is seen that to solve the vortex equation it is advisable to use the method of successive upper relaxation.

Let us describe in detail the algorithm for solving difference equations (7), (8) by the upper relaxation method.

First, equation (7) is reduced to the standard form

$$\begin{gathered}\bar{A}_i \omega_{i-1,j}^{k+1/2} - \bar{C}_i \omega_{i,j}^{k+1/2} + \bar{B}_i \omega_{i+1,j}^{k+1/2} = -\bar{F}_{i,j}^{k}, \\ i,j = 1,2,\dots,N-1 \;\; k = 0,1,\dots,M-1.\end{gathered} \qquad (18)$$

where

$$\bar{A}_i = 0,5\tau\left[\frac{\nu}{h^2} - \frac{1}{4h}\left(\psi_{i,j+1}^{k} - \psi_{i,j-1}^{k}\right)\right], \quad \bar{B}_i = 0,5\tau\left[\frac{\nu}{h^2} + \frac{1}{4h}\left(\psi_{i,j+1}^{k} - \psi_{i,j-1}^{k}\right)\right],$$

$$\bar{C}_i = 1 + \frac{\tau\nu}{h^2}, \qquad \bar{F}_{i,j}^{k} = \omega_{i,j}^{k} + \frac{0,5\tau\nu}{h^2}\left(\omega_{i,j+1}^{k} - 2\omega_{i,j}^{k} + \omega_{i,j-1}^{k}\right) +$$

$$+\frac{0,5\tau}{4h}\left(\omega_{i,j+1}^{k} - \omega_{i,j-1}^{k}\right)\left(\psi_{i+1,j}^{k} - \psi_{i-1,j}^{k}\right) + Q\left(t_{k+1/2}, x_i, y_i\right).$$

Difference equation (18) is solved by the sweep method, while $O(N^2)$ arithmetic operations are spent to determine the values of $\omega_{ij}^{k+1/2}$ at all nodes of the difference grid [13].

After all $\omega_{ij}^{k+1/2}$ are found, the difference equation (8) is solved, reducing it to the standard form:

$$\begin{gathered}A_j \omega_{i,j-1}^{k+1} - C_j \omega_{i,j}^{k+1} + B_j \omega_{i,j+1}^{k+1} = -F_{i,j}^{k+1/2}, \\ i,j = 1,2,\dots,N-1 \;\; k = 0,1,\dots,M-1.\end{gathered} \qquad (19)$$

where

$$A_j = 0.5\tau\left[\frac{\nu}{h^2} - \frac{1}{4h}\left(\psi_{i,j+1}^{k} - \psi_{i,j-1}^{k}\right)\right], \quad B_j = 0.5\tau\left[\frac{\nu}{h^2} + \frac{1}{4h}\left(\psi_{i,j+1}^{k} - \psi_{i,j-1}^{k}\right)\right],$$

$$C_j = 1 + \frac{\tau\nu}{h^2}, \quad F_{i,j}^{k+1/2} = \omega_{i,j}^{k+1/2} + \frac{0,5\tau\nu}{h^2}\left(\omega_{i+1,j}^{k+1/2} - 2\omega_{i,j}^{k+1/2} + \omega_{i-1,j}^{k+1/2}\right) +$$

$$+\frac{0,5\tau}{4h}\left(\omega_{i+1,j}^{k+1/2} - \omega_{i-1,j}^{k+1/2}\right)\left(\psi_{i,j+1}^{k} - \psi_{i,j-1}^{k}\right) + Q\left(t_{k+1}, x_i, y_j\right).$$

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To find all ω_{ij}^{k+1} by equation (19) by the sweep method, $O\left(N^2\right)$ arithmetic operations are required. For comparison, note that the solution of the two-dimensional implicit scheme for the vortex equation using the Gauss method requires $O\left(N^6\right)$ arithmetic operations.

Equations (15), (18), (19) are supplemented by the boundary conditions

$$\psi_{i,0}^{k+1}=0,\quad \psi_{i,N}^{k+1}=0,\quad i=0,1,...,N\;\; k=0,1,...,M-1. \tag{20}$$

$$\psi_{0,j}^{k+1}=0,\quad \psi_{N,j}^{k+1}=0,\quad j=0,1,...,N\;\; k=0,1,...,M-1. \tag{21}$$

and to determine the values of the vortex at the boundary grid nodes, Woods conditions are used [14]:

$$\omega_{i,0}^{k+1}+\frac{\omega_{i,1}^{k+1}}{2}=\frac{3\left(\psi_{i,0}^{k}-\psi_{i,1}^{k}\right)}{h^2},\qquad \omega_{i,N}^{k+1}+\frac{\omega_{i,N-1}^{k+1}}{2}=\frac{3\left(\psi_{i,N}^{k}-\psi_{i,N-1}^{k}\right)}{h^2}, \tag{22}$$

$$\omega_{0,j}^{k+1}+\frac{\omega_{1,j}^{k+1}}{2}=\frac{3\left(\psi_{0,j}^{k}-\psi_{1,j}^{k}\right)}{h^2},\qquad \omega_{N,j}^{k+1}+\frac{\omega_{N-1,j}^{k+1}}{2}=\frac{3\left(\psi_{N,j}^{k}-\psi_{N-1,j}^{k}\right)}{h^2}. \tag{23}$$

The systems of difference equations (18), (22) and (19), (23) are solved by the sweep method.

Let us present an algorithm for solving the boundary value problem (18), (22) by the sweep method:

$$\omega_{i,j}^{k+1/2}=\bar\alpha_{i+1}\omega_{i+1,j}^{k+1/2}+\bar\beta_{i+1},\; i=N-1,N-2,...,1,0,\;(0<j<N),\; k=0,1,...,M-1 \tag{24}$$

$$\bar\alpha_{i+1}=\frac{\bar B_i}{\bar C_i-\bar A_i\bar\alpha_i},\; \bar\beta_{i+1}=\frac{\bar A_i\bar\beta_i+\bar F_{ij}^k}{\bar C_i-\bar A_i\bar\alpha_i},\; i=1,2,...,N-1,\;\;(0<j<N),\; k=0,1,...,M-1. \tag{25}$$

$$\omega_{0,j}^{k+1/2}=\bar\alpha_1\omega_{1,j}^{k+1/2}+\bar\beta_1,\qquad \omega_{0,j}^{k+1/2}=-\frac{1}{2}\omega_{1,j}^{k+1/2}+\frac{3\left(\psi_{0j}^{k}-\psi_{1j}^{k}\right)}{h^2},$$
$$\bar\alpha_1=-0,5,\quad \bar\beta_1=\frac{3\left(\psi_{0j}^{k}-\psi_{1j}^{k}\right)}{h^2},\quad \left(0<j<N\right), \tag{26}$$

$$\omega_{N,j}^{k+1/2}+\frac{\omega_{N-1,j}^{k+1/2}}{2}=\frac{3\left(\psi_{N,j}^{k}-\psi_{N-1,j}^{k}\right)}{h^2}, \tag{27}$$

$$\omega_{N-1,j}^{k+1/2}=\bar\alpha_N\omega_{N,j}^{k+1/2}+\bar\beta_N, \tag{28}$$

Substituting (28) into (27), we obtain expressions for determining the value of $\omega_{N,j}^{k+1/2}$ at the boundary grid node, i.e. for $i = N$:

$$\omega_{N,j}^{k+1/2}=\left[\frac{3\left(\psi_{N,j}^{k}-\psi_{N-1,j}^{k}\right)}{h^2}-0,5\bar\beta_N\right]\Bigg/\left(1+0,5\bar\alpha_N\right), \tag{29}$$

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Now we present an algorithm for solving the boundary value problem (19), (23) by the sweep method:

$$\omega_{i,j}^{k+1}=\alpha_{j+1}\omega_{i,j+1}^{k+1}+\beta_{j+1},\quad j=N-1,N-2,...,1,0,\quad (0<i<N), \tag{30}$$

$$\alpha_{j+1}=\frac{B_j}{C_j-A_j\alpha_j},\quad \beta_{j+1}=\frac{A_j\beta_j+F_{i,j}^{k+1/2}}{C_j-A_j\alpha_j},\quad j=1,2,...,N-1,\ (0<i<N), \tag{31}$$

$$\begin{gathered}\omega_{i,0}^{k+1}=\alpha_1\omega_{i,1}^{k+1}+\beta_1,\quad \omega_{i,0}^{k+1}=-0,5\omega_{i,0}^{k+1}+3\left(\psi_{i,0}^{k}-\psi_{i,1}^{k}\right)\Big/h^2,\\ \alpha_1=-0,5,\quad \beta_1=3\left(\psi_{i,0}^{k}-\psi_{i,1}^{k}\right)\Big/h^2,\quad (0<i<N),\end{gathered} \tag{32}$$

$$\omega_{i,N}^{k+1}+0,5\omega_{i,N-1}^{k+1}=3\left(\psi_{i,N}^{k}-\psi_{i,N-1}^{k}\right)\Big/h^2, \tag{33}$$

$$\omega_{i,N-1}^{k+1}=\alpha_N\omega_{i,N}^{k+1}+\beta_N, \tag{34}$$

Substituting (34) into (33), we obtain expressions for determining the value of $\omega_{i,N}^{k+1}$ at the boundary node $j=N$:

$$\omega_{i,N}^{k+1}=\left[\frac{3\left(\psi_{i,N}^{k}-\psi_{i,N-1}^{k}\right)}{h^2}-0,5\beta_N\right]\Bigg/\left(1+0,5\alpha_N\right). \tag{35}$$

IV. CONCLUSION.

We present the results of numerical calculations for solving the Navier-Stokes equation based on the above methods.

To solve problem (7) - (12), the longitudinal-transverse method (Peaseman-Rackford and Douglas-Rackford schemes) and the iterative method of sequential upper relaxation were used. The grids are selected as follows: $h_x=h_y=0.05,\ \nu=1,\ \tau=0.001$. The computational experiment will be carried out by the trial function method. If the differential problem has an exact stationary solution $\psi(x,y)=\sin^2\pi x\sin^2\pi y$, then an expression for the function $Q(x,y)$ and $\omega(x,y)$ can be obtained [14]. From the current equation (2) we obtain the formulas for the function $\omega(x,y)$:

$$\omega(x,y)=-\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2}. \tag{36}$$

Find the derivatives

$$\frac{\partial^2\psi}{\partial x^2}=2\pi^2\cos 2\pi x\sin^2\pi y,\quad \frac{\partial^2\psi}{\partial y^2}=2\pi^2\cos 2\pi y\sin^2\pi x.$$

Supplying the found derivatives in (36), we obtain the following formulas for the function $\omega(x,y)$:

$$\omega(x,y)=2\pi^2\left(4\sin^2\pi x\sin^2\pi y-\sin^2\pi x-\sin^2\pi y\right).$$

In the stationary case, from the vortex equation (1) for $Q(x,y)$ we have

$$Q(x,y)=-\frac{\partial\omega}{\partial x}\frac{\partial\psi}{\partial y}+\frac{\partial\omega}{\partial y}\frac{\partial\psi}{\partial x}+\nu\left(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}\right). \tag{37}$$

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Putting (36) into (37), we obtain the following expression for $Q(x,y)$

$$Q(x,y)=\left(\frac{\partial^3\psi}{\partial x^3}+\frac{\partial^3\psi}{\partial x\partial y^2}\right)\frac{\partial\psi}{\partial y}-\left(\frac{\partial^3\psi}{\partial x^2\partial y}+\frac{\partial^3\psi}{\partial y^3}\right)\frac{\partial\psi}{\partial x}-$$
$$-\nu\left(\frac{\partial^4\psi}{\partial x^4}+2\frac{\partial^4\psi}{\partial x^2\partial y^2}+\frac{\partial^4\psi}{\partial y^4}\right). \qquad (38)$$

We put the partial derivatives of $\psi(x,y)$ in (38) and obtain the final expression for the function $Q(x,y)$:

$$Q(x,y)=8\pi^4\left\{8\sigma_1^2\sigma_2^2-3\sigma_1^2-3\sigma_2^2+1+\sigma_1\sigma_2\sigma_3\sigma_4\left[\sigma_2^2\left(4\sigma_3^2-3\right)-\sigma_1^2\left(4\sigma_4^2-3\right)\right]\right\},$$

$$\sigma_1=\sin\pi x,\ \ \sigma_2=\sin\pi y,\ \ \sigma_3=\cos\pi x,\ \ \sigma_4=\cos\pi y.$$

Figures 1 and 2 show the level lines of the stream function and vortex based on calculations of the exact solution and numerical results.

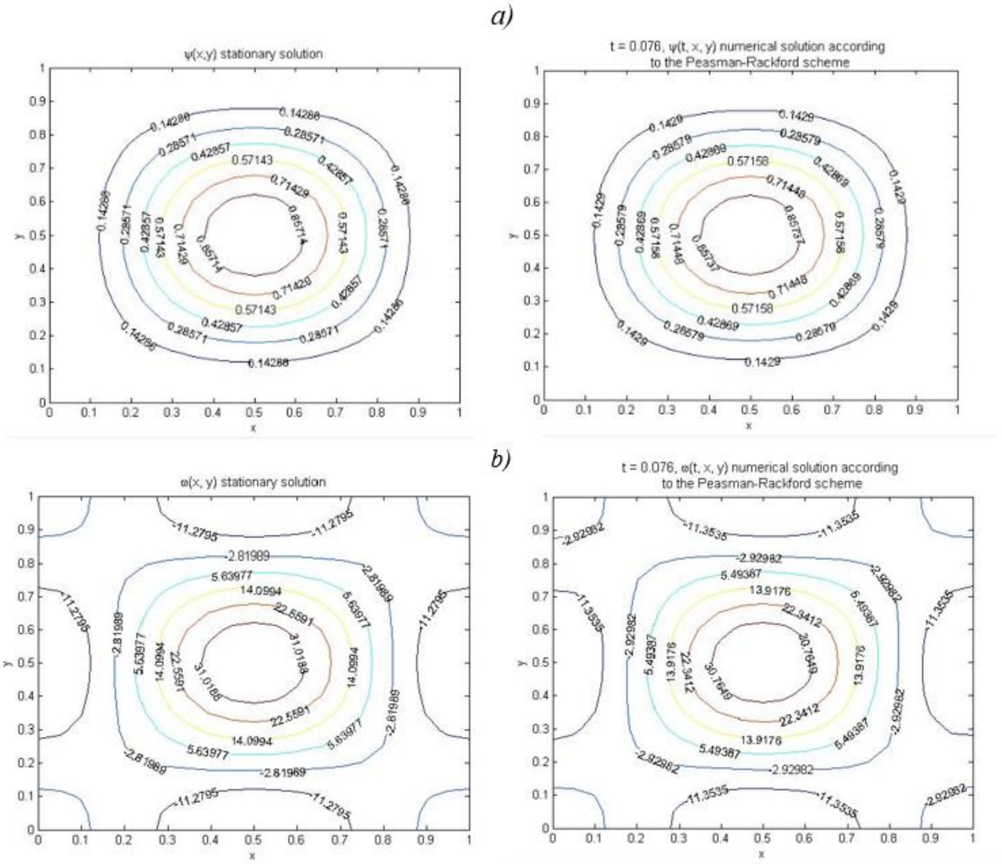

Figure 1. Stationary solution of the stream function $\psi(x,y)$ and the numerical solution of the stream function $\psi(t_k,x_i,y_j)$ according to the Peasman-Rackford scheme (*a*); stationary solution of the vortex function $\omega(x,y)$ and numerical solution of the vortex function $\omega(t_k,x_i,y_j)$ according to the Peasman-Rackford scheme (*b*).

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The methods of Peasman-Rackford and Douglas-Rackford were used to solve the problem. Comparison of the calculation results shows that the numerical solution obtained by the Peasman-Rackford method is close to the exact solution, in comparison with the Douglas-Rackford method. Therefore, for the numerical solution of the Navier-Stokes equation system, it is advisable to use the Peasman-Rackford method.

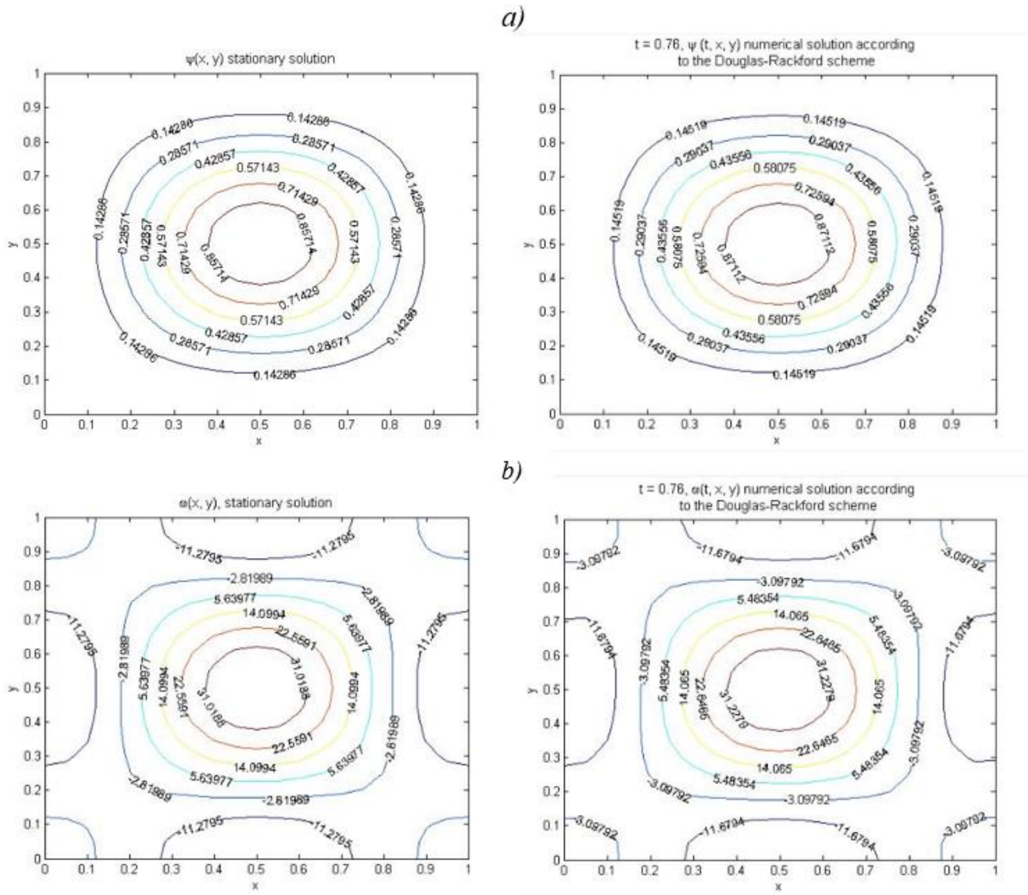

figure 2. stationary solution of the stream function $\psi(x,y)$ and the numerical solution of the stream function $\psi(t_k,x_i,y_j)$ according to the douglas-rackford scheme (*a*); stationary solution of the vortex function $\omega(x,y)$ and numerical solution of the vortex function $\omega(t_k,x_i,y_j)$ using the douglas-rackford scheme (*b*).

For an example of a nonstationary problem, consider equation (1) - (6) with a source $Q(t,x,y)$. It is necessary to choose $Q(t,x,y)$ so that the exact solution of the vortex equation has the form

$$\psi(t,x,y)=t\sin^2 2\pi x\sin^2 \pi y.$$

After simple operations from the equation of current and vortex using the selected exact solution, we obtain the exact solution of the vortex and the expression for $Q(t,x,y)$ in the following form:

$$Q(t,x,y)=8\pi^4 t\left(50\varphi_2^2\varphi_1-9\varphi_2^2-24\varphi_1+4\right)-2\pi^2\left(-10\varphi_2^2\varphi_1+\varphi_2^2+4\varphi_1\right)+$$

$$+32\pi^4t^2\varphi_3\cos\pi y\ \varphi_2\ \sin^3\pi y\left(5\varphi_3^2-3\right)+16\pi^4t^2\varphi_3\cos\pi y\ \varphi_2^3\sin\pi y\left(10\cos^2\pi y-9\right),$$

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$$\omega(x,y)=2\pi^2 t\left(10\sin^2 2\pi x\sin^2 \pi y-\sin^2 2\pi x+4\sin^2 \pi y\right),$$

$$\varphi_1=\sin^2 \pi y,\ \ \varphi_2=\sin 2\pi x,\ \ \varphi_3=\cos 2\pi x.$$

For the selected example, the Peasman-Rackford and Douglas-Rackford schemes are also used. The calculation results are shown in Figure 3 and 4.

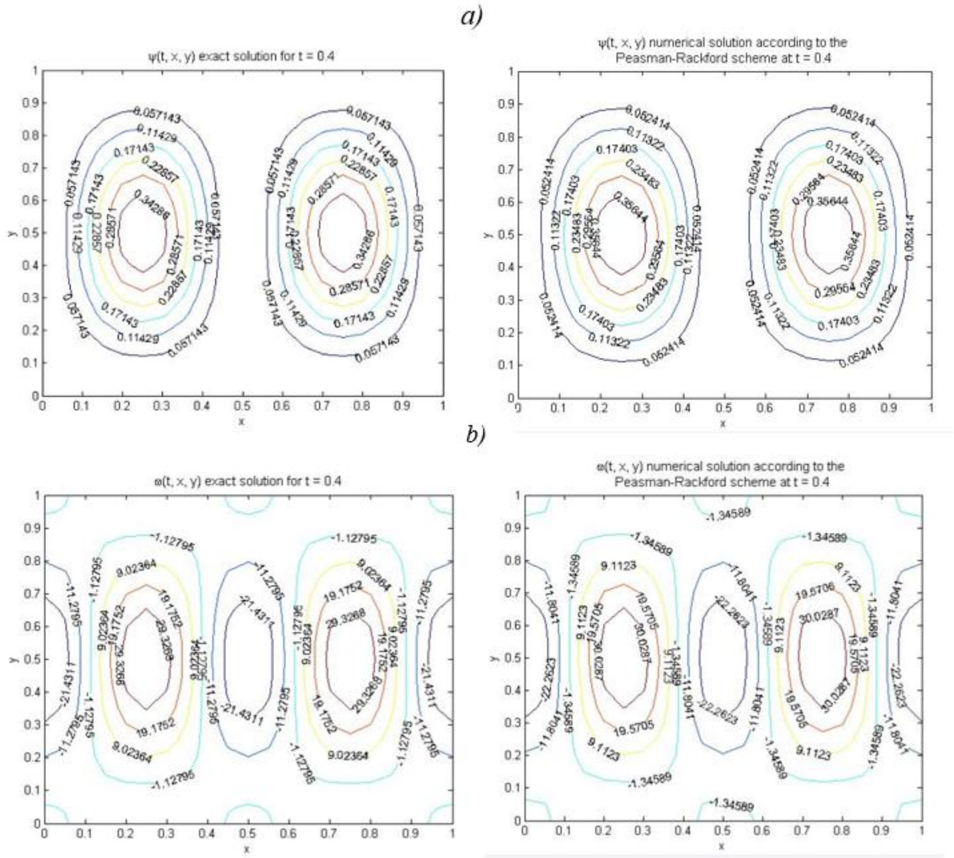

Figure 3. Exact solution of the stream function $\psi(t,x,y)$ and the numerical solution of the stream function $\psi(t_k,x_i,y_j)$ according to the Peasman-Rackford scheme (a); exact solution of the vortex function $\omega(t,x,y)$ and numerical solution of the vortex function $\omega(t_k,x_i,y_j)$ according to the Peasman-Rackford scheme (b).

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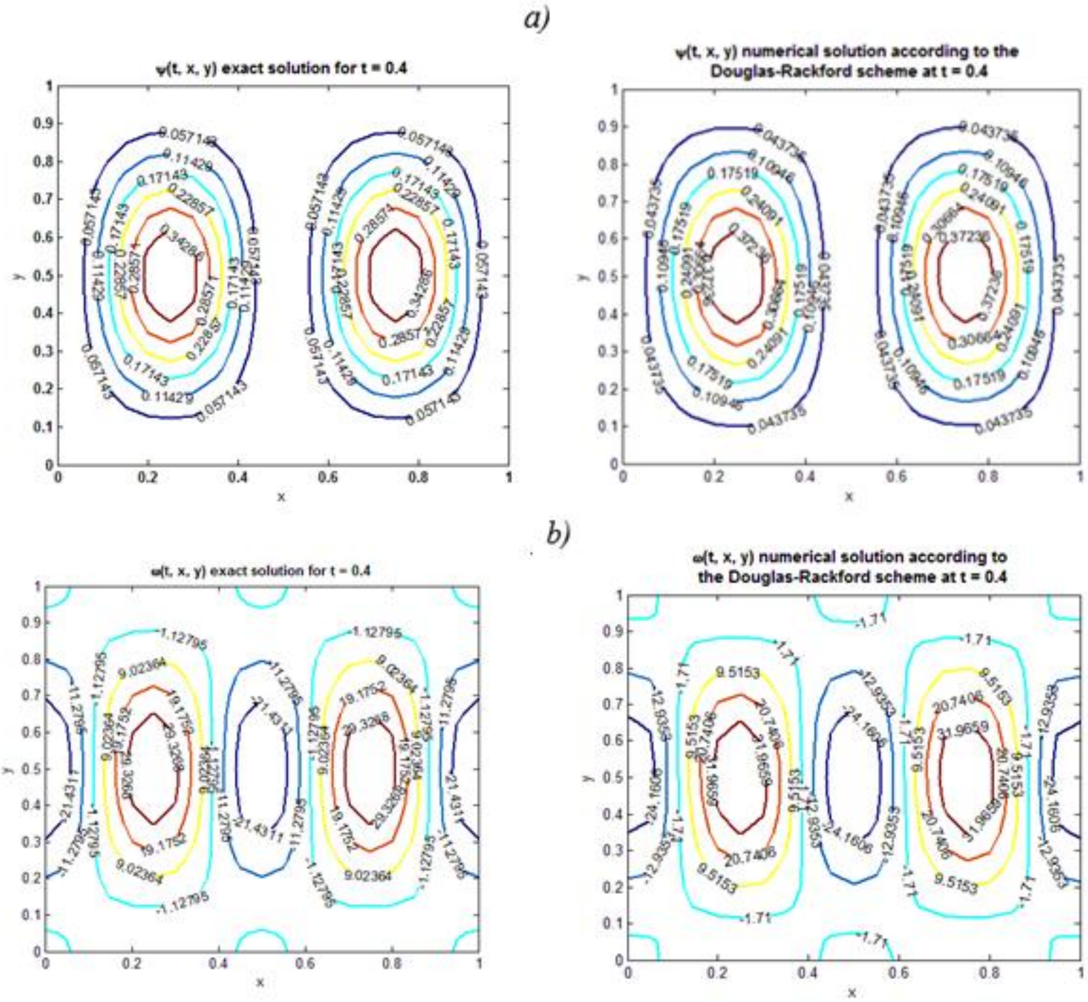

Figure 4. Exact solution of the stream function $\psi(t,x,y)$ and the numerical solution of the stream function $\psi(t_k,x_i,y_j)$ according to the Douglas-Rackford scheme (a); exact solution of the vortex function $\omega(t,x,y)$ and numerical solution of the vortex function $\omega(t_k,x_i,y_j)$ using the Douglas-Rackford scheme (b).

It can be seen from the numerical results that here it is also expedient to use the Peasman-Rackford schemes to obtain a numerical solution.

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