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Approximation of the Burgers Equation by Spectral-Grid Method

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ABSTRACT: The article deals with the numerical modeling of the Burgers equation describing nonlinear wave processes by the spectral-grid method. Since this equation is nonlinear, the solution of such equations requires the development of special economical methods. For this purpose, the spectral-grid method is used to solve the differential problem corresponding to the Burgers equation. In the region of integration, a grid is introduced, on each of the grid elements an approximate solution is sought in the form of a series in terms of the Chebyshev polynomial of the first kind. As a result, the solution to the problem is reduced to solving a system of algebraic-differential equations.

KEYWORDS:mathematical modeling, nonlinear wave systems, spectral - grid method, evolutionary problems, interval of integration, grid approximation, Chebyshev polynomials of the first kind, algebraic system, approximate solution, efficiency, numerical results.

I. INTRODUCTION

One of the most characteristic properties of wave movements is that they continue to exist even after the elimination of the causes that cause them [1]. Waves usually persist for a long time and can transmit disturbances over very long distances. In fact, waves acquire their most characteristic shape precisely after propagation to a "large" distance from the region in which they "originated". One of the important and difficult mathematical problems is related to the description of the behavior of waves of small amplitude, experiencing weak dissipation at large time intervals, which is described by the Burgers equation.

In [2], a numerical method is proposed for approximating the solution of the one-dimensional Burgers equation. Comparison results with exact values clearly show that the results obtained using the method are accurate and reliable.

In [3], two new methods are described that provide improved exponential finite-difference solutions of the Burgers equation. These methods are called the implicit exponential finite difference method and the fully implicit exponential finite difference method for solving the Burgers equation. The results are compared with the exact values and the results obtained using both methods are shown to be accurate and reliable.

The article [4] presents a new method for forming an improved exponential finite-difference solution of the Burgers equation. This method is called the Crank-Nicholson finite difference method for solving the Burgers equation. Comparison results with exact values show that the results obtained using the method are accurate and reliable.

The paper [5] proposes two meshless methods for solving the one-dimensional nonlinear inhomogeneous Burgers equation. These methods are based on schemes of a multi-quadratic quasi-interpolation operator and on networks of direct and indirect radial basis functions. In the presented schemes, the Taylor series expansion is used to discretize the time derivative, and quasi-interpolation is used to approximate the solution function and its spatial derivatives. Numerical solutions are compared with analytical solutions, as well as with the results of other numerical schemes.

In [6], an effective Galerkin finite element scheme for least squares is presented, which models the Burgers equation on the whole real line and is subject to initial conditions with compact support.

By directly comparing the numerical and analytical solutions and their asymptotic behavior, it was concluded that the proposed scheme is accurate even for large times, and it can be used to numerically study the properties of this and similar equations in unbounded domains.

The paper [7] presents new completely implicit numerical schemes for solving the one-dimensional and two-dimensional nonstationary Burgers equation. The non-linear Burgers equation is discretized in the spatial direction

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using the second-order finite difference method, which transforms the Burgers equation into a non-linear ordinary differential equation system. Comparison of numerical results with exact solutions and solutions of other schemes shows that the proposed schemes are simple, efficient, and accurate even for cases with high Reynolds numbers.

Self-oscillating systems occupy a special place among nonlinear systems [8]. Self-oscillating systems are clocks, lamp generators of electromagnetic oscillations, steam engines and internal combustion engines, in a word, all real systems that are capable of performing continuous oscillations in the absence of periodic external influences.

In [9-14], various aspects of the application of Chebyshev polynomials of the first kind in spectral methods for the numerical simulation of incompressible viscous flows are described.

Numerical solutions of the initial boundary value problems of hydro aerodynamics with large gradients encounter serious difficulties due to the lack of sufficient spatial resolution in regions of strong inhomogeneity. These difficulties in many cases are overcome by using spectral methods and their modifications, which have the property of high-precision spatial approximation with an increase in the number of basis functions. In applied calculations, however, it is not always possible to increase the number of basis functions to the required value, since the growth of the order of matrices in the resulting algebraic system is significantly limited by computer resources.

In this work, to overcome these difficulties, the spectral-grid method is used [15-19]. Depending on the type of initial data or the expected form of the solution, a grid is introduced in the integration interval. At the internal nodes of the grid, the requirement is imposed on the continuity of the approximate solution and its derivatives up to order $m-1$, where m is the order of the highest derivative of the differential equation. At the boundary grid nodes, the corresponding boundary conditions for the problem under consideration are set. An approximate solution on grid elements is represented in the form of finite series in Chebyshev polynomials of the first kind. The resulting system of equations with the help of linear non-degenerate transformations is reduced to two autonomous systems: a linear system of algebraic equations and a system (in the general case nonlinear) of ordinary differential equations. To solve the first system, standard methods are used, and to solve the second, an explicit algorithm is used, developed [20] and improved in this work.

Therefore, the use of the spectral-grid method makes it possible, firstly, to distribute the Chebyshev polynomials over the elements, taking into account the behavior of the solution gradient and, secondly, to lead to a significant decrease in the order of the matrices in the arising algebraic system. In this method, for a given number of grid elements N, to achieve the required accuracy of calculations, it is necessary to correctly position the grid nodes and select the number of polynomials p_j on the grid elements. These questions are closely related, because by bringing the grid nodes closer together, one can reduce the number of polynomials on the elements and vice versa. In practical calculations, it is more convenient to choose a uniform mesh, setting different number of polynomials p_j on each mesh element, or vice versa. Then the number of required polynomials depends on the relative magnitude of the gradients of the solution on a particular element. Solution gradients can often be estimated from asymptotic analysis. In problems with large gradients, it is known [21] that near the wall — in the so-called critical layer — the behavior of the solution is determined by a rapid change in viscous solutions; far from the wall, the perturbations slowly decay.

II. STATEMENT OF THE PROBLEM

One of the important and difficult mathematical problems is related to the description of the behavior of small-amplitude waves experiencing weak dissipation over long time intervals [1]. These restrictions are not as special as they might seem at first glance. Since, as follows from observations, waves can indeed exist for a long time outside the sources, the limitations associated with the assumption of low dissipation in large time intervals are quite natural. In gas dynamics, there is weak dissipation, characterized by the dimensionless parameter Re^{-1}, where Re is the Reynolds number. In this case, the value of the amplitude of wave ε is small, but finite. In cases of greatest interest, the corresponding nonlinear equation should be considered at time intervals of the order of ε^{-1}.

The studied process is described by the Burgers equation. We consider the following initial-boundary value problem

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$$\frac{\partial u}{\partial t}=\frac{1}{\mu}\frac{\partial^2 u}{\partial \eta^2}-u\frac{\partial u}{\partial \eta},a<\eta<b, \quad (1)$$

$$u(a,t)=0, \quad (2)$$

$$u(b,t)=0,$$

$$u(\eta,0)=u_0(\eta). \quad (3)$$

III. SPECTRAL-GRID METHOD

We divide the interval of integration $[a,b]$ into M different elements:

$$[\eta_0,\eta_1],[\eta_1,\eta_2],[\eta_2,\eta_3],\ldots,[\eta_{i-1},\eta_i],\ldots,[\eta_{M-1},\eta_M],$$

where $\eta_0=a$, $\eta_M=b$. To represent the approximate solution in the form of series in Chebyshev polynomials, each element $[\eta_{i-1},\eta_i]$ of integration interval $[a,b]$ is mapped to interval $[-1,1]$ using the following replacement of the independent variable

$$\eta_i=\frac{m_i}{2}+\frac{k_i}{2}y, \quad (4)$$

where $m_i=\eta_i+\eta_{i-1}$, $k_i=\eta_i-\eta_{i-1}$ - the length of the i th mesh element and $y\in[-1,1]$.After this transformation, problem (1) - (3) takes the form:

$$\frac{\partial u_i}{\partial t}=\frac{1}{\mu}\left(\frac{2}{k_i}\right)^2\frac{\partial^2 u_i}{\partial y^2}-\frac{2}{k_i}u_i\frac{\partial u_i}{\partial y},\ i=1,2,\ldots,M, \quad (5)$$

$$u_i(1)=u_{i+1}(-1),\ \ i=1,2,\ldots,M-1, \quad (6)$$

$$\frac{1}{k_i}\frac{\partial u_i}{\partial y}(1)=\frac{1}{k_{i+1}}\frac{\partial u_{i+1}}{\partial y},\ \ i=1,2,\ldots,M-1, \quad (7)$$

$$u_1(-1)=u_M(1)=0, \quad (8)$$

$$u_i(y,0)=u_0\left(\frac{m_i}{2}+\frac{k_i}{2}y,0\right),\ \ i=1,2,\ldots,M, \quad (9)$$

where Eqs. (6) - (7) require continuity of the approximate solution and its first derivative at the internal nodes of the grid, Eq. (8) is the form of the initial data (the initial data for the subsequent position are not of fundamental importance and therefore are not considered).

An approximate solution of the system of equations (5) - (8) will be sought in the form of series in the Chebyshev polynomials of the first kind $T_n(y)$ [15-19].

For this, we introduce matrix notation as follows. We replace in (5) - (7), taking into account $T_n(y)$ derivatives with respect to y expressions:

$$\frac{\partial v}{\partial y}=\hat{B}v, \quad (10)$$

$$\frac{\partial^2 v}{\partial y^2}=\hat{A}v, \quad (11)$$

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where v is a vector of length $(N+1)M$ with components:

$$v \equiv \{u_1(y_0)..u_1(y_N), u_2(y_0)..u_2(y_N), u_3(y_0)..., u_M(y_0)..u_M(y_N),\} \qquad (12)$$

in $\hat{A}$ and $\hat{B}$ - square matrices of dimension $((N+1)M)\times((N+1)M)$, having a block-diagonal structure:

$$\hat{A} = \begin{bmatrix} \hat{a}^1_{00} & \hat{a}^1_{01} & \cdots & \hat{a}^1_{0N-1} & \hat{a}^1_{0N} & & & & 0 & & & 0 & & & 0 & & \\ \hat{a}^1_{10} & \hat{a}^1_{11} & \cdots & \hat{a}^1_{1N-1} & \hat{a}^1_{1N} & & & & & & & & & & & & \\ \vdots & \vdots & \cdots & \vdots & \vdots & & & & & & & & & & & & \\ \hat{a}^1_{N-10} & \hat{a}^1_{N-11} & \cdots & \hat{a}^1_{N-1N-1} & \hat{a}^1_{N-1N} & & & & & & & & & & & & \\ \hat{a}^1_{N0} & \hat{a}^1_{N1} & \cdots & \hat{a}^1_{NN-1} & \hat{a}^1_{NN} & & & & & & & & & & & & \\ & & & & & \hat{a}^2_{00} & \hat{a}^2_{01} & \cdots & \hat{a}^2_{0N-1} & \hat{a}^2_{0N} & & & & & & & \\ & & & & & \hat{a}^2_{10} & \hat{a}^2_{11} & \cdots & \hat{a}^2_{1N-1} & \hat{a}^2_{1N} & & & & & & & \\ & & 0 & & & \vdots & \vdots & \cdots & \vdots & \vdots & 0 & & & 0 & & & \\ & & & & & \hat{a}^2_{N-10} & \hat{a}^2_{N-11} & \cdots & \hat{a}^2_{N-1N-1} & \hat{a}^2_{N-1N} & & & & & & & \\ & & & & & \hat{a}^2_{N0} & \hat{a}^2_{N1} & \cdots & \hat{a}^2_{NN-1} & \hat{a}^2_{NN} & & & & & & & \\ & & 0 & & & & & 0 & & & \ddots & & & 0 & & & \\ & & & & & & & & & & \hat{a}^M_{00} & \hat{a}^M_{01} & \cdots & \hat{a}^M_{0N-1} & \hat{a}^M_{0N} \\ & & & & & & & & & & \hat{a}^M_{10} & \hat{a}^M_{11} & \cdots & \hat{a}^M_{1N-1} & \hat{a}^M_{1N} \\ & & 0 & & & & & 0 & & & \vdots & \vdots & \cdots & \vdots & \vdots \\ & & & & & & & & & & \hat{a}^M_{N-10} & \hat{a}^M_{N-11} & \cdots & \hat{a}^M_{N-1N-1} & \hat{a}^M_{N-1N} \\ & & & & & & & & & & \hat{a}^M_{N0} & \hat{a}^M_{N1} & \cdots & \hat{a}^M_{NN-1} & \hat{a}^M_{NN} \end{bmatrix}$$

$$\hat{B} = \begin{bmatrix} \hat{b}^1_{00} & \hat{b}^1_{01} & \cdots & \hat{b}^1_{0N-1} & \hat{b}^1_{0N} & & & & 0 & & & 0 & & & 0 & & \\ \hat{b}^1_{10} & \hat{b}^1_{11} & \cdots & \hat{b}^1_{1N-1} & \hat{b}^1_{1N} & & & & & & & & & & & & \\ \vdots & \vdots & \cdots & \vdots & \vdots & & & & & & & & & & & & \\ \hat{b}^1_{N-10} & \hat{b}^1_{N-11} & \cdots & \hat{b}^1_{N-1N-1} & \hat{b}^1_{N-1N} & & & & & & & & & & & & \\ \hat{b}^1_{N0} & \hat{b}^1_{N1} & \cdots & \hat{b}^1_{NN-1} & \hat{b}^1_{NN} & & & & & & & & & & & & \\ & & & & & \hat{b}^2_{00} & \hat{b}^2_{01} & \cdots & \hat{b}^2_{0N-1} & \hat{b}^2_{0N} & & & & & & & \\ & & & & & \hat{b}^2_{10} & \hat{b}^2_{11} & \cdots & \hat{b}^2_{1N-1} & \hat{b}^2_{1N} & & & & & & & \\ & & 0 & & & \vdots & \vdots & \cdots & \vdots & \vdots & 0 & & & 0 & & & \\ & & & & & \hat{b}^2_{N-10} & \hat{b}^2_{N-11} & \cdots & \hat{b}^2_{N-1N-1} & \hat{b}^2_{N-1N} & & & & & & & \\ & & & & & \hat{b}^2_{N0} & \hat{b}^2_{N1} & \cdots & \hat{b}^2_{NN-1} & \hat{b}^2_{NN} & & & & & & & \\ & & 0 & & & & & 0 & & & \ddots & & & 0 & & & \\ & & & & & & & & & & \hat{b}^M_{00} & \hat{b}^M_{01} & \cdots & \hat{b}^M_{0N-1} & \hat{b}^M_{0N} \\ & & & & & & & & & & \hat{b}^M_{10} & \hat{b}^M_{11} & \cdots & \hat{b}^M_{1N-1} & \hat{b}^M_{1N} \\ & & 0 & & & & & 0 & & & \vdots & \vdots & \cdots & \vdots & \vdots \\ & & & & & & & & & & \hat{b}^M_{N-10} & \hat{b}^M_{N-11} & \cdots & \hat{b}^M_{N-1N-1} & \hat{b}^M_{N-1N} \\ & & & & & & & & & & \hat{b}^M_{N0} & \hat{b}^M_{N1} & \cdots & \hat{b}^M_{NN-1} & \hat{b}^M_{NN} \end{bmatrix}$$

$$\hat{A} = TPT^*, \quad \hat{B} = TRT^*. \qquad (13)$$

where T is the matrix of Chebyshev polynomials of the first kind, T^* are matrices formed by the inverse transformation of the Chebyshev matrix [21], P and R are matrices for the first and second spatial derivatives. T, T^*, P and R are block-diagonal matrices of dimension $((N+1)M)\times((N+1)M)$.

We also introduce matrices $\tilde{A}$ and $\tilde{B}$:

$$\tilde{A} = \frac{1}{\mu} K^2 \hat{A}, \quad \tilde{B} = K\hat{B}, \qquad (14)$$

where K is a diagonal matrix.

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Writing differential equations (5) only at the interior points of elements $(l=1,\ldots,N-1)$, conditions (6) and (7) at the boundary points of neighboring elements, and (8) at the boundary points of the interval, we arrive at the following system:

$$\frac{dS}{dt} = Av - v \cdot Bv, \tag{15}$$

$$Dv = 0. \tag{16}$$

Here, S denotes a vector of length $(N+1)M$:

$$S \equiv \{0, u_1(y_1)..u_1(y_{N-1}),0,0,u_2(y_1)..u_2(y_{N-1}),0,0, \\ u_3(y_1)...,0,u_M(y_1)..u_M(y_{N-1}),0\}, \tag{17}$$

matrices A, B of dimension $((N+1)M)\times((N+1)M)$ and matrix D of dimension $2M\times((N+1)M)$ [21].

As for matrix D, its first and last rows consist of the coefficients of conditions (8), in the remaining even rows $(l=2j,\ j=2,\ldots,M-1)$ are the coefficients of equations (6), and in odd rows $(l=2j-1,\ j=2,\ldots,M)$ are the coefficients of equations (7) (the corresponding rows of matrix $\tilde{B}$).

$$D = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & & & \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & & & & & & & & 0 & & \\
x & x & x & x & x & x & x & x & x & x & & & & & & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & & & & & \\
 & & & & & x & x & x & x & x & x & x & x & x & x & & & & & \\
 & & & & & & & & & & & & \ddots & & & & & & & \\
 & & & & & & & & & & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
 & & 0 & & & & & & & & x & x & x & x & x & x & x & x & x & x \\
 & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

System (15) - (16) is "differential-algebraic" - it contains $(N-1)M$ ordinary differential equations (15) and $2M$ linear algebraic conditions (16) with $(N+1)M$ unknowns. Let us show that it can be reduced to two autonomous systems of equations: a differential system of a smaller order $(N-1)M$ only at the interior points of the interval and an algebraic system of the standard form $Ax=b$ for the remaining components of the solution (at the boundary points and at the mesh nodes).

Let us denote by X, Y, V_j and W_j combinations of variables in conditions (15) and we will consider them as new dependent variables:

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$$V_j \equiv u_j(y_N) - u_{j+1}(y_0) = 0,\ j = 1,...,M-1$$
$$W_j \equiv \frac{1}{K_j}\frac{\partial u_j}{\partial y}(y_N) - \frac{1}{K_{j+1}}\frac{\partial u_{j+1}}{\partial y}(y_0) = 0,$$
$$j = 1,...,M-1 \qquad (18)$$
$$X \equiv u_1(y_0) = 0,$$
$$Y \equiv u_N(y_N) = 0,$$

then vector

$$w = \{X, u_1(y_1)..u_1(y_{N-1}), V_1, W_1, u_2(y_1)..u_2(y_{N-1}),$$
$$V_2, W_2, ..., V_{M-1}, W_{M-1}, u_M(y_1)..u_M(y_{N-1}), Y\}$$

coincides with S from (18) and is related to v from (12) by the relation:

$$w = Gv. \qquad (19)$$

Here G is a non-degenerate matrix of dimension $((N+1)M)\times((N+1)M)$ in which rows with numbers $j(N+1)$ $(j = 1,...,M-1)$ are rows of matrix D with numbers $2j$ $(j = 1,...,M-1)$, and rows $j(N+1)+1$ $(j = 1,...,M-1)$ are its rows with numbers $2j+1$.

We also introduce the notation:

$$\hat{H} = AG^{-1}, \qquad (20)$$
$$\hat{f} = v \cdot Bv, \qquad (21)$$

in which (15) takes the form:

$$\frac{dw}{dt} = \hat{H}w - \hat{f}, \qquad (22)$$

Thus, it can be seen from (22) that the system of ordinary differential equations (15) has been reduced to a system of lower dimension, and with a nondegenerate matrix H (all zero rows and all columns are excluded from matrix $\hat{H}$, the elements of which are multiplied by the corresponding zero elements of vector w).

To efficiently calculate the nonlinear term in equation (1), we introduce the following transformation:

$$v = G^{-1}w$$

Then formula (21) takes the form

$$\hat{f} = G^{-1}w \cdot BG^{-1}w,$$

Performing multiplication operations on matrices and vectors, we have:

$$\hat{f} = \bar{f} \cdot \tilde{f} = \{0, \bar{f}_1(y_1)...\bar{f}_1(y_{N-1}), 0, \bar{f}_2(y_0), \bar{f}_2(y_1)...\bar{f}_2(y_{N-1}), ...$$
$$, \bar{f}_M(y_0), \bar{f}_M(y_1)...\bar{f}_M(y_{N-1})\} \cdot \{0, \tilde{f}_1(y_1)...\tilde{f}_1(y_{N-1}), 0,$$
$$0, \tilde{f}_2(y_1)...\tilde{f}_2(y_{N-1}), .., 0, \tilde{f}_M(y_1)...\tilde{f}_M(y_{N-1})\} =$$
$$= \{0, \hat{f}_1(y_1)...\hat{f}_1(y_{N-1}), 0, 0, \hat{f}_2(y_1)...\hat{f}_2(y_{N-1}), .., 0, \hat{f}_M(y_1)...\hat{f}_M(y_{N-1})\}$$

Thus, equation (22) can be written as

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$$\frac{dr}{dt} = Hr - f\ , \tag{23}$$

where f denotes, vector $\hat{f}$ without zero components.

Here the dimension is $H : (M(N-1)) \times (M(N-1))$, and

$$r = \{u_1(y_1)..u_1(y_{N-1}), u_2(y_1)..u_2(y_{N-1}),.., u_M(y_1)..u_M(y_{N-1})\},$$
$$f = \{f_1(y_1)...f_1(y_{N-1}),\ f_2(y_1)...f_2(y_{N-1}),..,\ f_M(y_1)...f_M(y_{N-1})\}$$

- vectors of length $(M(N-1))$, moreover, r vector differs from vector v only by the absence of components numbered $(j-1)N+1$ and jN, where $j = 1,...,M$ .The missing components of vector v are found by solving the linear algebraic system (19).

The system of equations (23) is evolutionary. For its numerical solution in this work, as already mentioned, the method [16,17] was used. Namely, for the transition to the next time layer, an explicit scheme of the form was used:

$$r(t+\tau) = r(t) + RQr(t) - Rf(t),$$
$$R\varphi(t) = \frac{\tau}{12}\left[23\varphi(t) - 16\varphi(t-\tau) + 5\varphi(t-2\tau)\right], \tag{24}$$
$$Q = \frac{12}{\tau}\frac{\left(e^{H\tau} - E\right)}{\left(23E - 16e^{-H\tau} + 5e^{-2H\tau}\right)},$$

where matrix Q is obtained using special transformations of system (23) - see [20], R is the operator of the third-order Adams-Bashforth scheme, E is the unit matrix, τ is the integration step.

The use of scheme (24) in comparison with the usual Adams-Bashforth scheme makes it possible to significantly weaken the restrictions on τ associated with the requirements of its stability.

Thus, the sequence of calculations is as follows:

1) the region of integration is divided into a certain number of elements M ;
2) a one-step algorithm (for example, the Runge-Kutta method) finds a solution on the first two time layers: $t = \tau, t = 2\tau$;
3) a number of non-degenerate transformations of matrix H are carried out in order to weaken the stability condition;
4) according to the formula (24), the transition to a new time layer $t = \tau$ is carried out;
5) the components of the vector v at the boundary points of the elements are found from the solution of the algebraic system (19).

IV. RESULTS OF CALCULATIONS

The constructed algorithm was applied for numerical simulation of one-dimensional initial-boundary value problems for Burgers equations.

An initial-boundary value problem for the Burgers equation is considered:

$$\frac{\partial u}{\partial t} = \frac{1}{\mu}\frac{\partial^2 u}{\partial \eta^2} - u\frac{\partial u}{\partial \eta}, \tag{25}$$
$$u(\eta, 0) = -\sin \pi\eta,$$
$$u(\pm 1, t) = 0. \tag{26}$$

The analytical solution of problem (25) - (26) has the form of the ratio of series of Bessel functions [10] and has the following form:

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$$u(x,t)=4\pi v\left[\sum_{n=1}^{\infty} n a_n e^{-n^2\pi^2 tv}\sin n\pi x \Big/ \left(x_0+2\sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 tv}\cos n\pi x\right)\right],$$

where $a_n=(-1)^n I_n(1/2\pi v)$ and $I_n(z)$ denotes the modified Bessel functions of first kind.This analytical solution is numerically untractable at small $t\,(0\le t\le 2/\pi)$ and v as $I_z(z)$ with z going to infinity behaves asymptotically as $e^z(2\pi z)^{-1/2}$ independent of n.

For large numbers $\mu\left(\mu\ge 10^2\right)$, the comparison of the numerical solution of problem (25) - (26) with the analytical one is difficult due to the slow convergence of the series in the analytical solution [10]. Therefore, the accuracy of calculations was estimated by the order of magnitude of the last coefficients a_n^j in expansion (10) (the order of error does not exceed the indicated value for μ).

Specific calculations were carried out for the following parameter values: $\mu=\pi\cdot 10^2, n=64, \tau=10^{-2}/6\pi$. The results are shown in Figs. 1 and 2. In all figures, curves 1, 2, 3, 4 show the behavior of the numerical solution obtained by the spectral-grid method at time $t=0,0.2,0.5,1.0$, and curve 5 corresponds to time $t=2.0$. It can be seen in Fig. 1 $(M=1)$ that with an increase in the first derivative at point $\eta=0$, the numerical solution begins to oscillate and at $t=1.0$ the amplitude of the oscillations increases so much that not a single correct sign remains in the solution. Application of the spectral-grid method, i.e. in the case of dividing the region of integration into 2 elements: $[-1,0],[0,1]$ (Fig. 2), the amplitude of the oscillations at $t=0.5$ is significantly less - the solution has 2 correct signs. When the $t>0.5$ oscillations are smoothed out, the accuracy of the solution increases again - by time $t=2.0$, for example, it reaches $\varepsilon\sim 10^{-4}$.

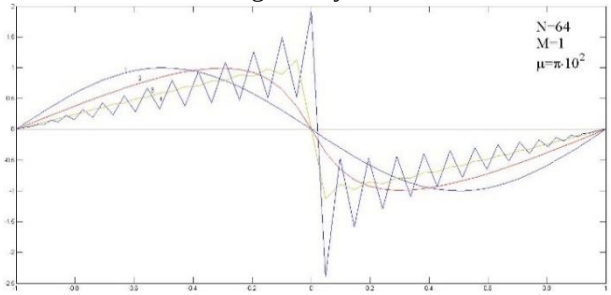

Fig. 1. Spectral method

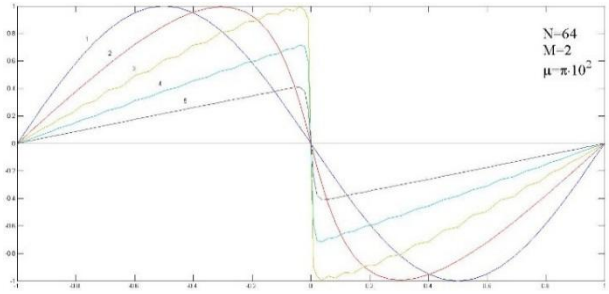

Fig. 2. Spectral–grid method

To increase the order of accuracy of calculations, we increase the number of grid elements in the spectral-grid method and consider the case of four elements, i.e. $M=4$.

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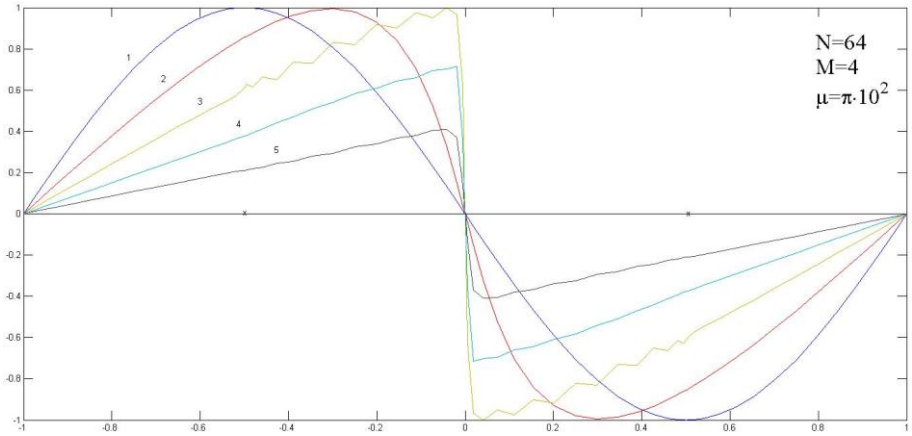

Fig. 3. Spectral–grid method

From the results of numerical calculations illustrated in Fig. 3, it can be seen that with an increase in the number of mesh elements, the accuracy increases. In accordance with this, we present the results on calculating the approximate solution in cases with 8 elements, i.e. when $M = 8$ (Fig. 4) with a uniform length of elements and with an equal number of approximating polynomials on the grid elements.

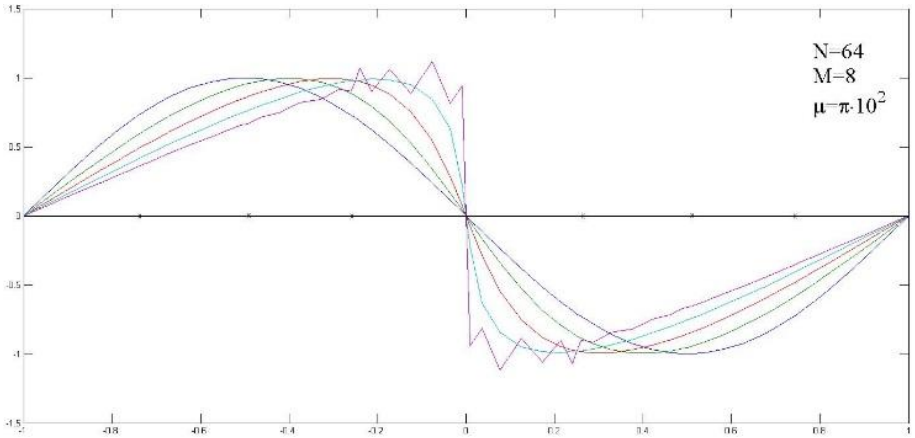

Fig. 4. Spectral–grid method

The calculation results show that increasing the number of grid elements will not lead to the expected results.

The approximate solution on the time interval $0.3 \le t < 0.5$ and at $-0.25 \le \eta \le 0.25$ in the spatial variable changes greatly and at $t = 0.5$ the behavior of the approximate solution becomes impossible to reflect with the existing capabilities of modern computers.

For these reasons, we increase the number of grid elements (respectively, the number of Chebyshev polynomials) precisely on interval $-0.25 \le \eta \le 0.25$. Thus, in the interval of integration $[-1,1]$, we introduce a grid consisting of four elements:

$$[-1,-0.2],[-0.2,0],[0,0.2],[0.2,1].$$

the total number of polynomials in all grid elements is equal to $N = 64$, 16 Chebyshev polynomials are selected on each grid element. The result is shown in Fig. 5.

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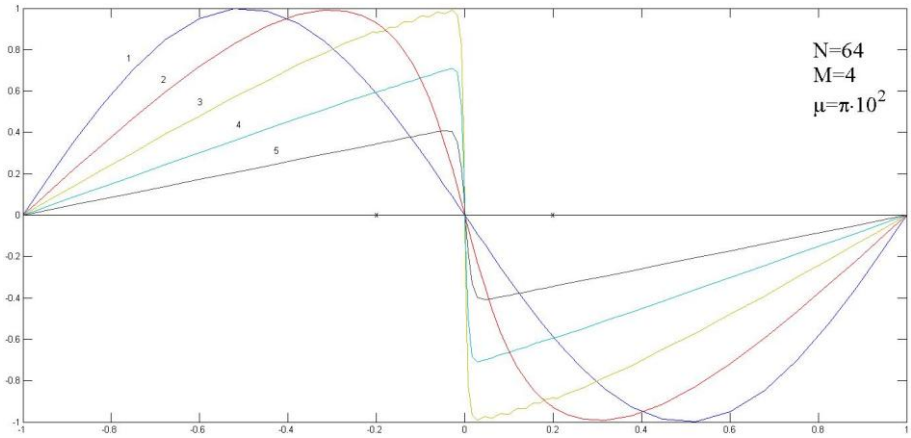

Fig. 5. Spectral–grid method

It can be seen that the oscillation of the approximate solution at the center of segment $[-1,1]$ is noticeably smoothed for all times t , i.e. the calculation accuracy is noticeably improved.

We continue to increase the grid elements at the center of segment $[-1,1]$, namely, consider $M=8$ elements that are symmetrical about the center of interval $[-1,1]$ (for example, on interval $[-1,0]$ the following intervals):

$$[-1,-0.47],[-0.47,-0.2],[-0.2,-0.07],[-0.07,0]$$

in this case, the total number of polynomials is still equal to $N=64$, on each of the elements 8 polynomials are selected. Graphical representations of approximate solutions for different values of time t are shown in Fig. 6.

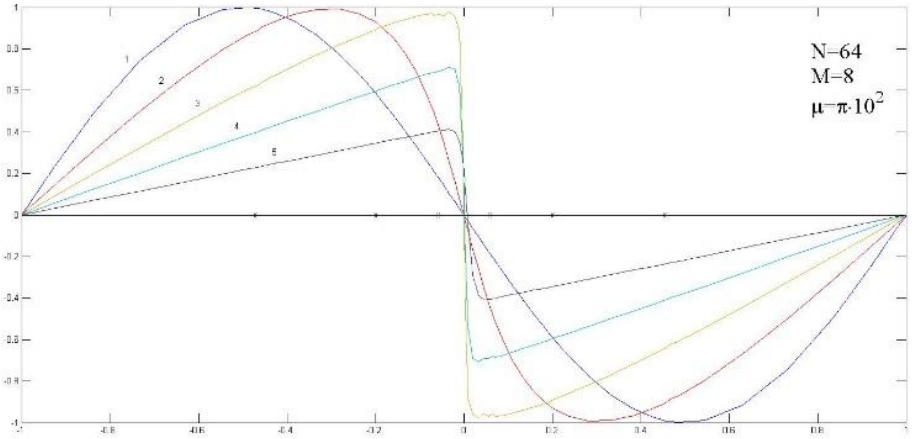

Fig. 6. Spectral-grid method.

It is seen that the oscillation of the approximate solutions is strongly smoothed for all values of the time t .

Thus, the calculation results show that due to the division of the integration interval into elements, i.e. When using the spectral-grid method, the accuracy of the approximate solution in areas with large gradients can be significantly increased.

Now, with the same partitioning of elements $M=8$, we increase the number of polynomials approximating the solution on the corresponding grid elements.

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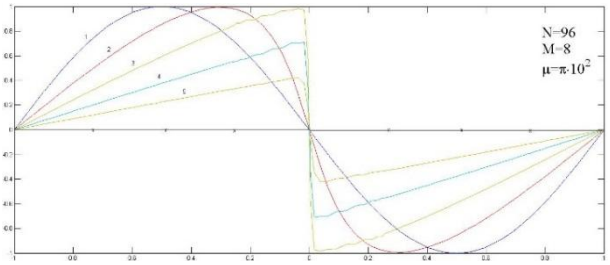

Fig. 7. Spectral grid method in a uniform grid.

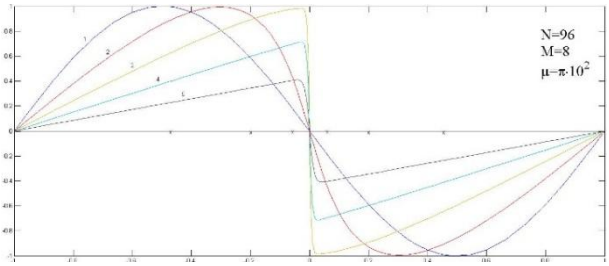

Fig. 8. Spectral grid method in non-uniform grid.

It is also of some interest to compare the spectral-grid method with the spectral-element method [20].Table 1 shows the value $\max\left|\frac{\partial u}{\partial \eta}\right|$ calculated by both methods, as well as the exact value of this value found from the analytical solution. The point in time at which the gradient of the solution reaches its maximum is designated here through $t_{\max}$.

Table1.

Calculation results ($\mu = 100\pi$)

Method	Interval	$\left\|\frac{\partial u}{\partial \eta}\right\|_{\max}$	$\pi t_{\max}$	$N \times M$	$\pi\tau$
1.Spektral-grid method (this work)	[-1,1]	152.4	1.6038	16×4	$10^{-2/6}$
2.Spectral-element method ([10])	[-1,1]	152.0	1.6033	16×4	$10^{-2/6}$
3.Analytical solution		152.0	1.6037		

It is seen that for both methods, the calculation accuracy is approximately the same order. The advantage of the spectral-grid method lies in its greater generality and efficiency, since, unlike [10], it does not contain any assumptions about the form of the solution, and therefore does not require the introduction of additional parameters determined in the calculation process. In addition, in the spectral-grid method, convenient recurrent formulas are used to organize computational procedures, while in the spectral-element method, to organize these procedures, one will have to deal with integral relations.

From the results of the calculations shown in Figures 5-6 it can be seen that the gradients of the solution vary greatly on the interval $[-0.2, 0.2]$, therefore, we consider it expedient to analyze the results precisely on this interval.

V. CONCLUSIONS

1. A spectral-grid method for solving evolutionary problems with large gradients has been constructed. Depending on the location of the regions of inhomogeneity, the integration interval is divided into a finite number of elements. At each of the elements, spectral approximation by finite series in basis functions is used.
2. Concrete calculations for the one-dimensional Burgers equation have been carried out, a comparison has been made with the one-element pseudo-spectral method.
3. It is shown that the use of the spectral-grid approximation makes it possible to significantly increase the accuracy of calculations without increasing the total number of basis functions.

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