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Applications of Fractional -Calculus to Certain Subclass of Analytic -Valent Functions with Negative Coefficients with TEBA operater

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ABSTRACT. In our paper we study a class $T(\alpha,\beta,b,\lambda,\mu)$,which consists of analytic and univalent functions with negative coefficients in the open unit disk U={z∈C:|z|<1}defined by Hadamard product (or convolution) with TEBA - Operator, we obtain coefficient bounds and extreme points for this class. Also distortion theorem using fractional calculus techniques and some results for this class are obtained.
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KEY WORDS AND PHRASES: Univalent Functin, Fractional Calculus, Hadamard Product, Distortion The orem TEBA-Operator,,Extreme Point..

The integral TEBA-operator of $f \in S$ for $\lambda > -1, \mu \geq 0$ is denoted by T_λ^μ and defined as following:

$$T_\lambda^\mu f(z) = \frac{(\lambda+1)^\mu}{\Gamma(\mu)} \int_0^1 t^\lambda \left(\log\frac{1}{t}\right)^{\mu-1} \frac{f(zt)}{t}\, dt = z - \sum_{n=2}^{\infty} \left(\frac{\lambda+1}{\lambda+n}\right)^{\mu} a_n z^n \quad (\lambda>-1, \mu\geq 0, f\in S)$$

(1)

The operator is known as the Komatu operator[2].A function $f \in S$, $z\in U$ is said to be in the class $T(\alpha,\beta,b,\lambda,\mu)$ if and only if it satisfies the inequality

$$\operatorname{Re}\left\{\beta\frac{T_\lambda^\mu f(z)}{z} + (1-\beta)(T_\lambda^\mu f(z))' + \alpha z (T_\lambda^\mu f(z))''\right\} > 1-|b| \qquad (2)$$

For some $\alpha(\alpha\geq 0), -1\leq\beta\leq 0, b\in\mathbf{C}, \lambda>-1$ and $\mu \geq 0$, for all $z\in U$.

The class $T(\alpha,0,1-\gamma,\lambda,0)$ was introduced Altintas[1] who obtained several results concerning this class .The class was $T(\alpha,0,b,\lambda,0)$ introduced by Srivastava and Owa[3]. The class $T(\alpha,\beta,b,\lambda,\mu)$ was introduced by Atshan and Kulkarni[1].

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Definition (1): We say that the function f of complex variable is analytic in a domain D if is differentiable at every point in that domain D.

Definition (2): A function f analytic in a domain D is said to be univalent there if it does not take the same value twice that is $f(z_1) \neq f(z_2)$ for all pairs of distinct points z_1 and z_2 in D.

In other word, f is one-to-one (or injective) mapping of D onto another domain. If $f(z)$ assumes the same value more than one, then f is said to be multivalent (p-valent) in D. Let A denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, n \in \mathbb{N} = \{1,2,3,\dots\} \qquad (1)$$

Which are analytic and univalent in the unit disk $U = \{z \in \mathbf{C} : |z| < \}$. If a function f is given by (1) and g is defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, n \in \mathbb{N} = \{1,2,3,\dots\} \qquad (2)$$

is in the class A, the convolution (or Hadamard product) of f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, z \in U \qquad (3)$$

Let S denote the subclass of A consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0 \qquad (4)$$

Definition (3)[4]: A function $f \in A$ is said to be starlike function of order α if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, (0 \leq \alpha < 1; z \in U) \qquad (5)$$

We denote the class of all starlike functions of order α in U by $S^*(\alpha)$.

Definition (4) [4]: A function $f \in A$ is said to be convex function of order α if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, (0 \leq \alpha < 1; z \in U) \qquad (6)$$

We denote the class of all convex functions of order α in U by $C(\alpha)$.

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Note that $S^*(0) = S^*$, $C(0) = C$ and $C \subset S^* \subset A$, and the Koebe function is starlike but not convex, where the Koebe function given by

$$K(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n$$

is the most famous function in the class A, which maps U onto **C** minus a slit along the negative real axis from $-\frac{1}{4}$ to $-\infty$.

Definition (5) [4]: A function f analytic in the unit disk U is said to be close-to-convex function of order $\alpha\ (0 \leq \alpha < 1)$ if there is a convex function g such that

$$\operatorname{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > \alpha, \quad \text{for all} \quad z \in U \qquad (7)$$

We denote by $K(\alpha)$ the class of close-to-convex functions of order α, f is normalized by the usual conditions $f(0) = f'(0) - 1 = 0$

These functions are connected by the relation $C \subset S^* \subset K$.

Definition (6) [7]: The fractional integral of order $\delta (0 < \delta)$ is defined by Where $f(z)$ is an analytic function in a simply connected region of Z-plane containing the origin and the multiplicity of $(z-t)^{\delta-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

Definition (7)[7]: The fractional derivative of order δ is defined by

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\delta}} dt, \qquad (9)$$

Where $f(z)$ is as in Definition (6) and the multiplicity of $(z-t)^{-\delta}$ is removed like Definition (6).

Definition (8)[7]:[Under the Condition of Definition(7)]

The fractional derivative of order $n + \delta\ (n = 0,1,2, \ldots)$ is defined by

$$D_z^{n+\delta} f(z) = \frac{d^n}{dz^n} D_z^{\delta} f(z)$$

From definition (1.1.6) and (1.1.7) by applying a simple calculation, we get

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(2+\delta)} z^{1+\delta} - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1+\delta)} a_n z^{n+\delta}, \qquad (10)$$

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$$D_z^{\delta} f(z)=\frac{1}{\Gamma(2-\delta)} z^{1-\delta}-\sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1-\delta)} a_n z^{n-\delta}. \qquad (11)$$

Definition(9)[4]: Let X be a topological vector space over the field of **C** and let E be a subset of X. A point $x \in E$ is called an extreme point of E if it has no representation of the form $x = ty + (1-t)z, 0 < t < 1,$ as a proper convex combination of two distinct points y and z in E.

Definition(10)[4]: Radius of starlikeness of a function f is the largest r_0 , $0 < r_0 < 1$ for which it is starlike in $|z|<r_0$.

Definition(11)[4]: Radius of convexity of a function f is the largest $r_1, 0 < r_1 < 1$ for which it is convex in $|z| < r_1$. .

Theorem (1)(Distortion Theorem[4]): For each $f \in A$

$$\frac{1-r}{(1+r)^3} \le |f'(z)| \le \frac{1+r}{(1-r)^3}, |z|=r<1$$

For each $z \in U, z \ne 0,$ equality occurs if and only if f is a suitable rotation of the Koebe function.

Theorem (2)(Growth Theorem[4]): For each $f \in A$

$$\frac{r}{(1+r)^2} \le |f(z)| \le \frac{r}{(1-r)^2}, |z|=r<1$$

For each $z \in U, z \ne 0$,equality occurs if and only if f is a suitable rotation of the Koebe function.

Lemma(1)(Schwarz Lemma): Let f be analytic in the unit disk U with $f(0) = 0$ and $|f(z)| < 1$ in U .Then $|f'(0)| \le 1$ and $|f(z)| \le |z|$ in U .Strict inequality holds in both estimates unless f is a rotation of the disk $f(z) = e^{i\theta} z$.

Theorem (2): Let the function f be in the class $T(\alpha, \beta, b, \lambda, \mu)$ .Then

$$\left|D_z^{-\delta} f(z)\right| \le \frac{1}{\Gamma(2+\delta)} |z|^{1+\delta} \left[1+\frac{2|b|}{(2-\beta+2\alpha)(2+\delta)\left(\frac{\lambda+1}{\lambda+2}\right)^{\mu}} |z| \right], \qquad (6)$$

and

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$$\left|D_z^{-\delta} f(z)\right| \geq \frac{1}{\Gamma(2+\delta)}|z|^{1+\delta}\left[1-\frac{2|b|}{(2-\beta+2\alpha)(2+\delta)\left(\frac{\lambda+1}{\lambda+2}\right)^{\mu}}|z|\right]. \qquad (7)$$

The inequalities in (6) and (7) are attained for the function

$$f(z)=z-\frac{|b|}{(2-\beta+2\alpha)\left(\frac{\lambda+1}{\lambda+2}\right)^{\mu}}z^2. \qquad (8)$$

Proof: Using Theorem(1) ,we have

$$\sum_{n=2}^{\infty} a_n \leq \frac{|b|}{(2-\beta+2\alpha)\left(\frac{\lambda+1}{\lambda+2}\right)^{\mu}}. \qquad (9)$$

From Definition (6), we have

$$D_z^{-\delta} f(z)=\frac{1}{\Gamma(2+\delta)}|z|^{1+\delta}-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+1+\delta)}a_n z^{n+\delta},$$

and

$$\Gamma(2+\delta)z^{-\delta}D_z^{-\delta} f(z)=z-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2+\delta)}{\Gamma(n+1+\delta)}a_n z^n=z-\sum_{n=2}^{\infty}\phi(n)a_n z^n, \qquad (2.10)$$

Where $\phi(n)=\frac{\Gamma(n+1)\Gamma(2+\delta)}{\Gamma(n+1+\delta)}$.

We know that $\phi(n)$ is a decreasing function of n and

$$0<\phi(n)\leq\phi(2)=\frac{2}{2+\delta}.$$

Using (9) and (10), we have

$$\left|\Gamma(2+\delta)z^{-\delta}D_z^{-\delta} f(z)\right| \leq |z|+\phi(2)|z|^2\sum_{n=2}^{\infty}a_n \leq |z|+\frac{2|b|}{(2-\beta+2\alpha)\left(\frac{\lambda+1}{\lambda+2}\right)^{\mu}(2+\delta)}|z|^2,$$

Which gives (6), we also have

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$$\left|\Gamma(2+\delta)z^{-\delta}D_z^{-\delta}f(z)\right|\geq|z|-\phi(2)|z|^2\sum_{n=2}^{\infty}a_n\geq|z|-\frac{2|b|}{(2-\beta+2\alpha)\left(\frac{\lambda+1}{\lambda+2}\right)^{\mu}(2+\delta)}|z|^2$$

Which gives (7). This complete the proof

Theorem(3): Let the function f be in the class $T(\alpha,\beta,b,\lambda,\mu)$ .Then

$$\left|D_z^{\delta}f(z)\right|\leq\frac{1}{\Gamma(2-\delta)}|z|^{1-\delta}\left[1+\frac{2|b|}{(2-\beta+2\alpha)\left(\frac{\lambda+1}{\lambda+2}\right)^{\mu}(2-\delta)}|z|\right],\quad(11)$$

and

$$\left|D_z^{\delta}f(z)\right|\geq\frac{1}{\Gamma(2-\delta)}|z|^{1-\delta}\left[1-\frac{2|b|}{(2-\beta+2\alpha)\left(\frac{\lambda+1}{\lambda+2}\right)^{\mu}(2-\delta)}|z|\right].\qquad(12)$$

The inequalities in (11) and (12) are attained for the function f given by (8)

Proof: Using Theorem (2), we have

$$\sum_{n=2}^{\infty}a_n\leq\frac{2|b|}{(2-\beta+2\alpha)\left(\frac{\lambda+1}{\lambda+2}\right)^{\mu}}\qquad(13)$$

By definition (7), we get

$$D_z^{\delta}f(z)=\frac{1}{\Gamma(2-\delta)}|z|^{1-\delta}-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+1-\delta)}a_nz^{n-\delta}$$

and

$$\Gamma(2-\delta)z^{\delta}D_z^{\delta}f(z)=z-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}a_nz^n$$

$$=z-\sum_{n=2}^{\infty}\frac{n!\Gamma(2-\delta)}{\Gamma(n+1-\delta)}a_nz^n=z-\sum_{n=2}^{\infty}\psi(n)a_nz^n,\qquad(14)$$

Since

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$\psi(n)=\frac{n!\Gamma(2-\delta)}{\Gamma(n+1-\delta)}$ is decreasing function of n and

$0<\psi(n)\le\psi(2)=\frac{1}{2-\delta}$, using (13) and (14), we have

$$\left|\Gamma(2-\delta)z^{\delta}D_z^{\delta}f(z)\right|\le|z|+\psi(2)|z|^2\sum_{n=2}^{\infty}a_n\le|z|+\frac{2|b|}{(2-\beta+2\alpha)\left(\frac{\lambda+1}{\lambda+2}\right)^{\mu}(2-\delta)}|z|^2$$

Which gives (11); and

$$\left|\Gamma(2-\delta)z^{\delta}D_z^{\delta}f(z)\right|\ge|z|-\psi(2)|z|^2\sum_{n=2}^{\infty}a_n\ge|z|-\frac{2|b|}{(2-\beta+2\alpha)\left(\frac{\lambda+1}{\lambda+2}\right)^{\mu}(2-\delta)}|z|^2$$

Which gives (12).
Now, we concentrate upon getting the radius of close-to-convexity ,starlikeness and convexity

Theorem(4): If $f\in T(\alpha,\beta,b,\lambda,\mu)$, then f is close-to-convex of order ε in $|z|<r_1(\alpha,\beta,b,\lambda,\mu,\varepsilon)$, where

$$r_1(\alpha,\beta,b,\lambda,\mu,\varepsilon)=\inf_n\left\{\frac{(1-\varepsilon)(\beta+n(1-\beta+\alpha n-\alpha))\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu}}{n|b|}\right\}^{\frac{1}{n-1}}.$$

Proof: It is sufficient to show that

$$|f'(z)-1|=\left|-\sum_{n=2}^{\infty}na_nz^{n-1}\right|<\sum_{n=2}^{\infty}na_n|z|^{n-1}\le1-\varepsilon \qquad (15)$$

and

$$\sum_{n=2}^{\infty}[\beta+n(1-\beta+\alpha n-\alpha)]\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu}a_n\le|b| \qquad (16)$$

Observe that (15) is true if

$$\frac{n|Z|^{n-1}}{1-\varepsilon}\le\frac{(\beta+n(1-\beta+\alpha n-\alpha))\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu}}{|b|} \qquad (17)$$

Solving (17) for $|z|$, we obtain

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$$|z| \le \left[\frac{(1-\varepsilon)(\beta+n(1-\beta+\alpha n-\alpha))\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu}}{n|b|} \right]^{\frac{1}{n-1}}, n=2,3,\ldots \;.$$

This completes the proof.

Theorem (5): If $f \in T(\alpha,\beta,b,\lambda,\mu)$, then f is starlike of order ε in $|z|<r_2(\alpha,\beta,b,\lambda,\mu,\varepsilon)$, where

$$r_2(\alpha,\beta,b,\lambda,\mu,\varepsilon)=\inf_n \left\{ \frac{(1-\varepsilon)(\beta+n(1-\beta+\alpha n-\alpha))\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu}}{(n-\varepsilon)|b|} \right\}^{\frac{1}{n-1}} \quad n=2,3,\ldots \;.$$

Proof: We must show that

$$\left|\frac{zf'(z)}{f(z)}-1\right|<1-\varepsilon, \quad \text{for} \quad |z|<r_2(\alpha,\beta,b,\lambda,\mu,\varepsilon) \;.$$

Since

$$\left|\frac{zf'(z)}{f(z)}-1\right|=\left|\frac{zf'(z)-f(z)}{f(z)}\right|=\left|\frac{z-\sum_{n=2}^{\infty} na_n z^n - z+\sum_{n=2}^{\infty} a_n z^n}{z-\sum_{n=2}^{\infty} a_n z^n}\right| \le \frac{\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty} a_n|z|^{n-1}},$$

(18)

by using (16), we observe (18) less than or equal $1-\varepsilon$ if

$$\frac{(n-\varepsilon)|z|^{n-1}}{1-\varepsilon} \le \frac{(\beta+n(1-\beta+\alpha n-\alpha))\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu}}{|b|} \qquad (19)$$

Solving (19) for $|z|$, we obtain

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$$|z| \le \left[\frac{(1-\varepsilon)(\beta+n(1-\beta+\alpha n-\alpha))\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu}}{(n-\varepsilon)|b|} \right]^{\frac{1}{n-1}}, n=2,3,\ldots .$$

This completes the proof.

Theorem (6): If $f \in T(\alpha,\beta,b,\lambda,\mu)$, then f is convex of order ε in $|z|<r_3(\alpha,\beta,b,\lambda,\mu,\varepsilon)$, where

$$r_3(\alpha,\beta,b,\lambda,\mu,\varepsilon)=\inf_n \left\{ \frac{(1-\varepsilon)(\beta+n(1-\beta+\alpha n-\alpha))\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu}}{n(n-\varepsilon)|b|} \right\}^{\frac{1}{n-1}} .$$

Proof: If is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left| \frac{-\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1-\sum_{n=2}^{\infty} na_n z^{n-1}} \right| \le \frac{\sum_{n=2}^{\infty} n(n-1)a_n |z|^{n-1}}{1-\sum_{n=2}^{\infty} na_n |z|^{n-1}} \le 1-\varepsilon \qquad (20)$$

By using (16), we observe that (20) is true if

$$\frac{n(n-\varepsilon)|z|^{n-1}}{1-\varepsilon} \le \frac{(\beta+n(1-\beta+\alpha n-\alpha))\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu}}{|b|}, \qquad (21)$$

solving (21) for $|z|$, we obtain

$$|z| \le \left[\frac{(1-\varepsilon)(\beta+n(1-\beta+\alpha n-\alpha))\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu}}{n(n-\varepsilon)|b|} \right]^{\frac{1}{n-1}}, n=2,3,\ldots .$$

This complete the proof.

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