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Fekete-Szegö estimate for a Class of Convex Functions Involving Certain Analytic Multiplier Transform

Deborah O. Makinde,Sh.Najafzadeh,T.O. Opoola

Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Nigeria,
Department of Mathematics, Payame Noor University, Tehran, Iran,
Department of Mathematics, University of Ilorin, Ilorin, Nigeria.

ABSTRACT: In this paper, we investigated the initial coefficient estimates and the Fekete-Szegö problem for a subclass of analytic univalent functions involving the linear transformation $D^{s}_{\alpha,\beta,\gamma}f$ for the normalized analytic univalent functions f of the form

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots$$

KEY WORDS: Analytic, univalent, starlike, linear transformation, coefficient estimates, Fekete-Szegö inequality. MSC[2010]: 30C45

I. INTRODUCTION AND PRELIMINARIES

Let A be the class of functions analytic in the unit disk $U = \{z : |z| < 1\}$ and S the subclass of A, which consist of functions of the form

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots, a_n \in \mathbb{C} \quad (1)$$

Satisfying $f(0) = f' - 1 = 0$in U.

For the class of normalized analytic and univalent functions S, Fekete and Szegö[1], proved that,

$$\phi_f(\lambda) = |a_3 - \lambda a_2^2| \leq 1 + 2_e - \frac{2\lambda}{1-\lambda}, \qquad 0 < \lambda \leq 1 \quad (2)$$

Kanas and Darwish [2] remarked that, when $\lambda = 1$ in equation (2), we have, $\phi f = a_3 - a_2^2$, which is equivalent to $\frac{S_f(0)}{6}$, where S_f denotes the Schwarzian derivative which is given by

$$S_f = \left(\frac{f''}{f}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 \quad (3)$$

and that if we consider the nth root transformation

$$(f(z^n))^{\frac{1}{n}} = z + c_{n+1}z^{n+1} + c_{2n+1}z^{2n+1} + \cdots$$

of the functions in equation (1), then$c_{n+1} = \frac{a2}{2}$ and $c_{2n+1} = \frac{a3}{n} + \frac{(n-1)a_2^2}{2n^2}$, so that

$$a_3 - \lambda a_2^2 = n(c2_n + 1 - \mu c_n^2 + 1)$$

where $\mu = \lambda n + (n-1)/2$. In [3] Makinde et al remarked that, Several authors have discussed the nature of $\phi_f(\lambda)$ for classes of normalized univalent functions in the unit disk and this is known as Fekete-Szegö problem. For instance in [4], Choi, Kim and Sugawa gave a generalized of Fekete-Szegöproblem for prestarlike functions, while in [5] Fekete-szeg ö problem was solved using subordination principle. Moreover, in [6], [7], [8], [11] and [12] Fekete-Szegöproblems were solved for class of close-to-convex functions. Authors in [8], [10], [13] and [14] also solved Fekete-szegöfor classes of normalized analytic functions.

Now, let $S^*(\beta), S^c(\beta)$ be the classes of starlike and convex univalent functions of orderβ, of the form

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$$S^{*} = \left\{ f \in S: Re\left(\frac{zf'(z)}{f(z)}\right) > \beta, \quad ,0 \le \beta < 1, \quad z \in U \right\} \tag{4}$$

$$S^{c}\left\{ f \in S: \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta, \ 0 \le \beta < 1, \quad z \in U \right\}, \tag{5}$$

Several authors have generalized notions of $\beta -$ starlikeness onto a complex order β see [15], [16], [17]. When$\beta = 0$ in equations (4) and (5), the starlike, respectively, convex functions with respect to the origin is obtained. With the aid of Ruscheweyh derivative, for the functions $f \in S$, Kumar et al [18] solved the Fekete-szegő problem for the class of analytic functions of complex order of the form $S_n(b)$ satisfying

$$Re\left\{1 + \frac{1}{b}\left(\frac{z(D^n f)'(z)}{D^n f(z)} - 1\right)\right\} > 0 \tag{6}$$

and Makinde et al in [3] solved the Fekete-szegő problem for a class of starlike functions using certain analytic multiplier transform. Remark 1 When $n = 0$ in *eq.(6),we have the class of starlike functions of order* $1 - b$.

Motivated by the work of Kanas and Darwish, using the linear transformation of Makinde et al inthe subclass of Kumar et al , we study the coefficient estimates and solve the Fekete-szegő problem for the subclass of convex functions $S_n^c(b)$.

Definition 1 Let b *be a nonzero complex number, and* f *univalent functions of the form (1). We say that* f *belongs to*$S_n^c(b)$ *if*

$$Re\left\{1 + \frac{1}{b}\left(\frac{z(F^n)''(z)}{F^n(z)'}\right)\right\} > 0, z \in U, \tag{7}$$

where$F = D^s_{\alpha,\beta,\gamma} f$ *is as given in equation (7)*

The following results shall be employed in the proof of the main results of this study.

Lemma 1 [20]: *Let* P *be the class of analytic functions in* U *with*$P(0) = 1, Re\, p(z) > 0$*and of the form*

$$p(z) = 1 + c1z + c2z^2 + \cdots, \tag{8}$$

then

$$|c_n| \le 2, n \ge 1.$$

If $|c_1| < 2$ *then* $p(z) \equiv p_1 = \frac{(1+\gamma 1 z)}{(1-\gamma 1 z)}$*with* $\gamma_1 = \frac{c1}{2}$. *Conversely, if* $p(z) \equiv p_1$ *for some* $\gamma_1 = 1$, *then* $c1 = 2_{\gamma_1}$ *and* $|c_1| = 2$. *Furthermore, we have*

$$\left| c_2 - \frac{c_1^2}{2} \right| \le 2 - \frac{|c_1|^2}{2}$$

If $|c_1| < 2$ *and* $\left|c_2 - \frac{c_1^2}{2}\right| = 2 - \frac{|c_1|^2}{2}$, *then* $p(z) \equiv p_2$, *where*

$$p_2(z) = \frac{1 + z\dfrac{\gamma 2z + \gamma 1}{1 + \gamma 1\, \gamma 2^z}}{1 - z\dfrac{\gamma 2^z + \gamma 1}{1 + \gamma 1 \gamma 2^z}}$$

and $\gamma_1 = \frac{c_1}{2}, \gamma_2 = \frac{2c2 - c_1^2}{4 - |c_1|^2}$.*Conversely, if* $p(z) = p2$ *for some* $\gamma 1 < 1$ *and* $\gamma_2 = 1$, *then* $\gamma 1 = \frac{c_1}{2}, \gamma_2 = \frac{2_{c2} - c^2}{4 - |c_1|^2}$*and* $\left|c_2 - \frac{c_1^2}{2}\right| = 2 - \frac{|c_1|^2}{2}$.

In what follows, we give the statement and proof of the results of this study.

II. COEFFICIENT ESTIMATES FOR THE SUBCLASS $S_n^c(b)$

Theorem 1:*Let* $n \in \mathbb{N}_0$*and* b *a non-zero complex number. If* f *of the form (1) is in* $S_n^c(b)$, *then*

$$|a_2^i| \le |b| \left(\frac{\alpha + \beta + \gamma}{\alpha + 2\beta + 4\gamma}\right)^s$$

and

$|a_3^i| \le \frac{|b|}{3}\left(\frac{\alpha+\beta+\gamma}{\alpha+3\beta+9\gamma}\right)^s max[1, |1 + 2b|], \beta, \gamma \ge 0; \alpha \ge 1; \ge s \in \mathbb{N}_0, 1 \le i \le k.$

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Proof *Let* $f \in S_n^c(b)$, *then by definition 2, there exist an analytic functions* p *given by*

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (9)$$

Satisfying $P(0) = 1$ *and* $Re(p(z)) > 0$ *such that*

$$1 + \frac{1}{b}\left(\frac{z(F^n)''(z)}{(F^n)'(z)}\right) = p(z) \quad (10)$$

Where $F = D^S_{\alpha\beta\gamma}$

From equation (10), we have:

$$\frac{z(F^n)''(Z)}{(F^n)'(Z)} = b(p(z) - 1) \quad (11)$$

Equating coefficients in equation (11) using equations (8), we have

$$a_2 = \frac{1}{2} t_2^{-s} b c_1 \quad (12)$$

And

$$a_3 = \frac{1}{6} t_3^{-s} b[c_2 + c_1^2 b] \quad (13)$$

Where $t_2 = \left(\frac{\alpha+2\beta+4\gamma}{\alpha+\beta+\gamma}\right)$ *and* $t_3 = \left(\frac{\alpha+3\beta+9\gamma}{\alpha+\beta+\gamma}\right)$.

Using equations (12) and(13) with lemma 1, we have

$$|a_2| \le t_2^{-s}|b|$$

And

$$|a_3| = \left|\frac{1}{6} t_3^{-s} b[c_2 + c_1^2 b]\right|$$

$$= \left|\frac{1}{6} t_3^{-s} b\left[c_2 - \frac{c_1^2}{2} + \frac{1+2b}{2} c_1^2\right]\right|$$

$$\le \frac{|b|}{6} t_3^{-s}\left[2 - \frac{|c1|^2}{2} + \frac{|1+2b|}{2}|c1|^2\right]$$

$$= \frac{|b|}{6} t_3^{-s}\left[2 + \frac{c1^2}{2}(|1+2b| - 1)\right]$$

$$\frac{b}{3} t_3^{-s} \max[1, |1+2b|]$$

Which proofs theorem 1.

III THE FEKETE-SZEGő PROBLEM FOR THE SUBCLASSES $S_n^c(b)$

Theorem 2 *Let* b *be a non-zero complex number and* $.f \in S_n^c(b)$.
Then, for $\mu \in \mathbb{C}$, *the following holds.*

$$|a_3 - \mu a_2^2| \le \frac{1}{3} t_3^{-s}|b| \max\left\{1, \left|1 + 2b - 3b\mu t_2^{-s}\frac{(\alpha+3\beta+9\gamma)^s}{(\alpha+2\beta+4\gamma)^{2s}}\right|\right\}$$

Proof From equations (12) and (13), we have

$$a_3 - \mu a_2^2 = \frac{b}{6} t_3^{-s}[c_2 + bc_1^2] - \mu(bt_2^{-s}c_1)^2$$

$$= \frac{b}{6} t_3^{-s}\left[c_2 + bc_1^2 - \frac{1}{4}\mu b^2 t_2^{2s} c_1^2\right]$$

$$= \frac{b}{6} t_3^{-s}\left[c_2 + c_1^2 b - \frac{3}{2}\mu t_2^{-s}\left(\frac{\alpha+3\beta+9\gamma}{\alpha+2\beta+4\gamma}\right) bc_1^2\right]$$

$$= \frac{b}{6} t_3^{-s}\left[c_2 - \frac{c_1^2}{2} + \frac{c_1^2}{2}\left(1 + 2b - 3b\mu t_2^{-s}\left(\frac{\alpha+3\beta+9\gamma}{\alpha+2\beta+4\gamma}\right)\right)\right]$$

$$\le \frac{b}{6} t_3^{-s}\left[2 - \frac{|c1|^2}{2} + \frac{|c1|^2}{2}\left|1 + 2b + 3b\mu t_2^{-s}\left(\frac{\alpha+3\beta+9\gamma}{\alpha+2\beta+4\gamma}\right)\right|\right]$$

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$$= \frac{|b|}{6} t_3^{-s} \left[2 + \frac{|c_1|^2}{2} \left(\left| 1 + 2b - 3b\mu t_2^{-s} \left(\frac{\alpha + 3\beta + 9\gamma}{\alpha + 2\beta + 4\gamma} \right) \right| - 1 \right) \right]$$

$$\leq \frac{|b|}{3} t_3^{-s} \max \left\{ 1, \left| 1 + 2b - 3b\mu t_2^{-s} \left(\frac{\alpha+3\beta+9\gamma}{\alpha+2\beta+4\gamma} \right) \right| \right\}$$

$$Where\ t_2 = \left(\frac{\alpha+2\beta+4\gamma}{\alpha+\beta+\gamma} \right) and\ t_3 = \left(\frac{\alpha+3\beta+9\gamma}{\alpha+\beta+\gamma} \right).$$

This proofs the theorem.

Theorem 3 Let b*be a non-zero complex number and* $f \in S_n^c(b)$.

Then, for $\mu \in \mathbb{R}$, *the following holds.*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \left|\frac{b}{3}\right| t_3^{-s} \left[\left| 1 + 2b \left(1 - \frac{3}{2} \mu t_2^{-s} \left(\frac{\alpha + 3\beta + 9\gamma}{\alpha + 2\beta + 4\gamma} \right) \right) \right| \right] & if\ \mu \leq \frac{3}{2} t_2^s \left(\frac{\alpha + 2\beta + 4\gamma}{\alpha + 3\beta + 9\gamma} \right) \\ \frac{|b|}{3} t_3^{-s} & if\ \frac{3}{2} t_2^s \left(\frac{\alpha + 2\beta + 4\gamma}{\alpha + 3\beta + 9\gamma\gamma} \right) \leq \mu \leq \frac{2}{3|b|} t_2^s \left(\frac{\alpha + 2\beta + 4\gamma}{\alpha + 3\beta + 9\gamma} \right) \\ \frac{|b|}{3} t_3^{-s} \left[\left| 2b \left(\mu \frac{(\alpha + 3\beta + 9\gamma)^s}{(\alpha + 2\beta + 4\gamma)^{2s}} - 1 \right) - 1 \right| \right] & if\ \mu \geq \frac{2}{3|b|} t_2^s \left(\frac{\alpha + 2\beta + 4\gamma}{\alpha + 3\beta + 9\gamma} \right) (2|b| + 1) \end{cases} \quad (14)$$

Where $t_2 = \left(\frac{\alpha+2\beta+4\gamma}{\alpha+\beta+\gamma} \right)$, $t_3 = \left(\frac{\alpha+3\beta+9\gamma}{\alpha+\beta+\gamma} \right)$

Proof 3 *Let* $\mu \leq \frac{3}{2} t_2^s \left(\frac{\alpha+3\beta+9\gamma}{\alpha+2\beta+4\gamma} \right)$ *From equation (17),*

we have

$$|a_3 - \mu a_2^2| \leq |b| t_2^{-s} \left[\left| 1 + 2b(1 - \mu \frac{(\alpha + 3\beta + 9\gamma)^s}{(\alpha + 2\beta + 4\gamma)^{2s}}) \right| \right]$$

Now, using the above calculations with

$\frac{3}{2} t_2^s \left(\frac{\alpha+2\beta+4\gamma}{\alpha+3\beta+9\gamma} \right) \leq \mu \leq \frac{2}{3|b|} t_2^s \left(\frac{\alpha+2\beta+4\gamma}{\alpha+3\beta+9\gamma} \right) (2|b| + 1)$, *we have*

$a_3 - \mu a_2^2 \leq |b| t_3^{-s}$

And conclusively, let $\mu \geq \frac{2}{3|b|} t_2^s \left(\frac{\alpha+2\beta+4\gamma}{\alpha+3\beta+9\gamma} \right) (2|b| + 1)$, *then*

$$|a_3 - \mu a_2^2| \leq \frac{|b|}{2} t_2^{-s} \left[2 + \frac{|c_1|^2}{2} \left(2\mu|b| \frac{(\alpha + 3\beta + 9\gamma)}{(\alpha + 2\beta + 4\gamma)^{2s}} - 2 - 2|b| \right) \right]$$

$$\leq |b| t_3^{-s} \left[2\mu|b| \frac{(\alpha + 3\beta + 9\gamma)^s}{(\alpha + 2\beta + 4\gamma)^{2s}} - 1 - 2|b| \right]$$

This concludes the proof of the theorem 3.

IV.CONCLUSION

The results in this paper extend the work of Kanas and Darwish as it is evident that for $s = 1$ and $\alpha + \beta + \gamma = \frac{\alpha+2\beta+4\gamma}{2}$, $s = 0$ in the first part, respectively second part of the theorem 1 yields the first part respectively second part of the theorem 2.5 for $n = 0$ Kanasand Darwish. Moreover, for$n > 1$, a refinement initial coefficient estimates were obtained.

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