ISSN: 2350-0328
International Journal of Advanced Research in Science, Engineering and Technology
Vol. 6, Issue 10 , October 2019

Fractional Calculus of a class of Univalent Functions with Some Geometric Properties

Karrar khudhair obayes , Aseel Ameen Harbi, Halah Mohammed Abass, Hala Abbas Mehdi

College of Computer Science and Information Technology, University of Al-Qadisiyah
College of Computer Science and Information Technology, University of Al-Qadisiyah
College of Computer Science and Information Technology, University of Al-Qadisiyah
College of Computer Science and Information Technology, University of Al-Qadisiyah

ABSTRACT: In our paper we study a class $AH(\alpha,\beta,b,\lambda,\mu)$,which consists of analytic and univalent functions with negative coefficients in the open unit disk U={z∈C:|z|<1}defined by Hadamard product (or convolution) with HARBI - Operator, we obtain coefficient bounds and extreme points for this class. Also distortion theorem using fractional calculus techniques and some results for this classare obtained.
2000 Mathematics Subject Classifications: 30C45

KEY WORDSAND PHRASES: Univalent Functin, Fractional Calculus, Hadamard Product, Distortion The orem HARBI-Operator,,Extreme Point..

I.INTRODUCTION

The integral HARBI-operator of $f \in S$ for $\lambda > -1, \mu \geq 0$ is denoted byH_λ^μ and defined as following:

$$H_\lambda^\mu f(z) = \frac{(\lambda+1)^\mu}{\Gamma(\mu)} \int_0^1 t^\lambda \left(\log\frac{1}{t}\right)^{\mu-1} \frac{f(zt)}{t} dt = z - \sum_{n=2}^{\infty} \left(\frac{\lambda+1}{\lambda+n}\right)^{\mu} a_n z^n (\lambda > -1, \mu \geq 0, f \in S) \quad (1)$$

The operator is known as the Komatu operator[2].A function $f \in S$, $z \in U$ is said to be in the class $AH(\alpha,\beta,b,\lambda,\mu)$ if and only if it satisfies the inequality

$$\operatorname{Re}\left\{\beta \frac{H_\lambda^\mu f(z)}{z} + (1-\beta)(H_\lambda^\mu f(z))' + \alpha z (H_\lambda^\mu f(z))''\right\} > 1 - |b| \quad (2)$$

For some $\alpha(\alpha \geq 0), -1 \leq \beta \leq 0, b \in \mathbb{C}, \lambda > -1$ and$\mu \geq 0$, for all $z \in U$.

The class $AH(\alpha, 0, 1-\gamma, \lambda, 0)$ was introduced b Altintas[1] who obtained several results concerning this class .The class $AH(\alpha, 0, b, \lambda, 0)$ was introduced by Srivastava and Owa[3].
The class $AH(\alpha, \beta, b, \lambda, 0)$ was introduced by Atshan and Kulkarni[1].
Definition (1): We say that the function f of complex variable is analytic in a domain D if is differentiable at every point in that domain D.
Definition (2): A function f analytic in a domain D is said to be univalent there if it does not take the same value twice that is $f(z_1) \neq f(z_2)$ for all pairs of distinct points z_1 and z_2 in D. **Definition (3):** A function $f \in A$ is said to be convex function of order α if and only if

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha,(0\le\alpha<1;z\in U)$$

We denote the class of all convex functions of order α in U by$C(\alpha)$.
Note that $S^*(0) = S^*$, $C(0) = C$ and $C \subset S^* \subset A$, and the Koebe function is starlike but not convex, where the Koebe function given by

$$K(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n$$

is the most famous function in the class A , which maps U onto **C** minus a slit along the negative real axis from $-\frac{1}{4}$ to $-\infty$

2-Main Results

In the following theorem, we derive the coefficient inequality for the class $AH(\alpha, \beta, b, \lambda, \mu)$.

Theorem (1): Let $f \in S$ .Then f is in the class $AH(\alpha, \beta, b, \lambda, \mu)$ if and only if

$$\sum_{n=2}^{\infty}[\beta+n(1-\beta+\alpha n-\alpha)]\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu}a_n\le|b| \qquad (3)$$

The result (3) is sharp.

Proof: Assume that $f \in AH(\alpha,\beta,b,\lambda,\mu)$ .Then, we find from (.2) that

$$\mathrm{Re}\left\{\beta\left[1-\sum_{n=2}^{\infty}a_n\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu}z^{n-1}\right]+(1-\beta)\left[1-\sum_{n=2}^{\infty}na_n\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu}z^{n-1}\right]\right.$$

$$\left.+\alpha z\left[-\sum_{n=2}^{\infty}n(n-1)a_n\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu}z^{n-2}\right]\right\}>1-|b|.$$

If we choose z to be the real and let $z\to1$, we get

$$1-\sum_{n=2}^{\infty}[\beta+n(1-\beta+\alpha n-\alpha)]\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu}a_n\ge1-|b|,$$

Which is equivalent to (3).conversely, assume that (3) is true. Then, we have

$$\left|\beta\frac{H_{\lambda}^{\mu}f(z)}{z}-(1-\beta)(H_{\lambda}^{\mu}f(z))'-\alpha z(H_{\lambda}^{\mu}f(z))''-1\right|$$

$$\leq \sum_{n=2}^{\infty}[\beta+n(1-\beta+\alpha n-\alpha)]\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu} a_n \leq |b|.$$

The implies that $f \in AH(\alpha,\beta,b,\lambda,\mu)$. The result (3) is sharp for the function

$$f(z)=z-\frac{|b|}{[\beta+n(1-\beta+\alpha n-\alpha)]\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu}} z^n, n\geq 2 \quad (4)$$

In the following theorem, we obtain interesting properties of the class $f \in AH(\alpha,\beta,b,\lambda,\mu)$.

Theorem (2): Let $f \in AH(\alpha,\beta,b,\lambda,\mu)$ .Then

$$|z|-\frac{|b|}{2-\beta+2\alpha}|z|^2 \leq \left|Q_{\lambda}^{\mu} f(z)\right| \leq |z|+\frac{|b|}{2-\beta+2\alpha}|z|^2 \quad (5)$$

Proof: It is easy to see that , for $f \in AH(\alpha,\beta,b,\lambda,\mu)$,

$$(2-\beta+2\alpha)\sum_{n=2}^{\infty}\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu} a_n \leq \sum_{n=2}^{\infty}[\beta+n(1-\beta+\alpha n-\alpha)]\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu} a_n \leq |b|.$$

Hence

$$\sum_{n=2}^{\infty}\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu} a_n \leq \frac{|b|}{(2-\beta+2\alpha)}, 2-\beta+2\alpha \leq \beta+n(1-\beta+\alpha n-\alpha)$$

Now,

$$\left|H_{\lambda}^{\mu} f(z)\right| = \left|z-\sum_{n=2}^{\infty}\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu} a_n z^n\right| \leq |z|+\sum_{n=2}^{\infty}\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu} a_n |z|^n$$

$$\leq |z|+|z|^2\sum_{n=2}^{\infty}\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu} a_n \leq |z|+|z|^2\frac{|b|}{(2-\beta+2\alpha)},$$

and

$$\left|H_{\lambda}^{\mu} f(z)\right| = \left|z - \sum_{n=2}^{\infty}\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu} a_n z^n\right| \ge |z| - \sum_{n=2}^{\infty}\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu} a_n |z|^n$$

$$\ge |z| - |z|^2 \sum_{n=2}^{\infty}\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu} a_n \ge |z| - |z|^2 \frac{|b|}{(2-\beta+2\alpha)}.$$

Theorem(3): Let $f(z) = z - \sum_{n=2}^{\infty} a_{n,i} z^n (a_n, i \ge 0, i = 1,2,\dots,m)$

be in the class $AH(\alpha,\beta,b,\lambda,\mu)$ .Then the function

$$K(z) = \sum_{i=1}^{m} d_i f_i(z) \ , (\sum_{i=1}^{m} d_i = 1)$$

is in the class $AH(\alpha,\beta,b,\lambda,\mu)$.

Proof: By definition of $K(z)$, we have

$$K(z) = z - \sum_{n=2}^{\infty}\left[\sum_{i=1}^{m} d_i a_{n,i}\right] z^n$$

Thus, we have from Theorem(.1)

$$\sum_{n=2}^{\infty}[\beta + n(1-\beta+\alpha n-\alpha)](\frac{\lambda+1}{\lambda+n})^{\mu}\left[\sum_{i=1}^{m} d_i a_{n,i}\right]$$

$$= \sum_{i=1}^{m} d_i \left[\sum_{n=2}^{\infty}[\beta + n(1-\beta+\alpha n-\alpha)](\frac{\lambda+1}{\lambda+n})^{\mu} a_{n,i}\right] \le \sum_{i=1}^{m} d_i |b| = |b|,$$

Which completes the proof of Theorem (3?)

REFERENCES

[1 W.G. Atshan and S. R. Kulkarni, On a new class of analytic functions with negative coefficients, Analele Universitatii Oradea Fasc. Matematica, TomXVI (2009), 43-51.
[2] Y. Komatu, On analytic prolongation of a family of operators , Mathematica (Cluj) 39 (55) (1990) , 141-145.
[3]H.M. Srivastava and S. Owa (Eds.), Current Topics in Analytic Function Theory , World Scientific , Singapore , 1992.