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On a Subclass of Multivalent Functions Defined by Differential Operator

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ABSTRACT: Making use of a differential operator, we introduce a new subclass $T(\lambda,\beta,m,n,p,j)$ of multivalent analytic functions in the open unit dick U. We study coefficient inequality, radii of close-to-convexity, starlikeness and convexity, integral mean for functions belonging to the defined class are obtained and we discuss some classes preserving integral operators.

KEY WORDS: Multivalent function, differential operator, convolution, radii of close-to-convexity, integral mean.

I.INTRODUCTION

Let $W(p,j)$ be the class of all functions of the form:

$$f(z)=z^p+\sum_{k=j+p}^{\infty}a_k z^k, \quad (k\geq j+p;\ p,j\in\mathbb{N}=\{1,2,\dots\}), \tag{1}$$

which are analytic and multivalent in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$.
Let $T(p,j)$ denote the subclass of $W(p,j)$ containing of functions of the form:

$$f(z)=z^p+\sum_{k=j+p}^{\infty}a_k z^k, \quad (a_k\geq 0;\ k\geq j+p;\ p,j\in\mathbb{N}=\{1,2,\dots\}), \tag{2}$$

which are analytic and multivalent in the open unit disk U.
If $f\in T(p,j)$ is given by (2) and $g\in T(p,j)$ given by

$$g(z)=z^p+\sum_{k=j+p}^{\infty}b_k z^k, \quad (k\geq j+p;\ p,j\in\mathbb{N}=\{1,2,\dots\}), \tag{3}$$

then the Hadamard product (or convolution) $f*g$ of f and g is defined by

$$(f*g)(z)=z^p+\sum_{k=j+p}^{\infty}a_k b_k z^k=(g*f)(z). \tag{4}$$

A function $f\in T(p,j)$ is said to be p-valently starlike of order ρ if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\}>\rho, \quad (0\leq\rho<p;\ z\in U). \tag{5}$$

A function $f\in T(p,j)$ is said to be p-valently convex of order ρ if

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\rho, \quad (0\leq\rho<p;\ z\in U). \tag{6}$$

It follows from expression (5), (6) that $f(z)$ is convex if and only if $zf'(z)$ is starlike.
A function $f\in T(p,j)$ is close-to-convex of order ρ if

$$Re\left\{\frac{f'(z)}{z^{p-1}}\right\}>\rho, \quad (0\leq\rho<p;\ z\in U). \tag{7}$$

For a function $f(z)$ in the class $T(p,j)$, Aouf and Mostafa [2] defined the differential operator $D_p^n: T(p,j)\to T(p,j)$, where

$$D_p^0 f(z)=f(z),$$
$$D_p^1 f(z)=D_p f(z)=\frac{z}{p}f'(z),$$

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$$D_p^2 f(z) = D\left(D_p f(z)\right),$$

$$D_p^n f(z) = D\left(D_p^{n-1} f(z)\right) = z^p + \sum_{k=j+p}^{\infty} \left(\frac{k}{p}\right)^n a_k z^k, \qquad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

For $p = j = 1$, the differential operator D^n was introduced by Salagean [8].

Definition (1): A function $f \in T(p,j)$ is said to be in the class $T(\lambda,\beta,m,n,p,j)$ if and only if

$$\left| \frac{(1-\lambda)z\left(D_p^n f(z)\right)'' + \lambda z\left(D_p^{n+m} f(z)\right)''}{(1-\lambda)\left(D_p^n f(z)\right)' + \lambda\left(D_p^{n+m} f(z)\right)'} - (p-1) \right| < \beta, \tag{8}$$

where $0 \le \lambda \le 1, 0 < \beta \le 1, (m,n \in \mathbb{N}_0), (m,p,j \in \mathbb{N})$ and $z \in U$.
Some of the following properties studied for other classes in [1,3,4,5]

II. COEFFICIENT INEQUALITY

In the following theorem, we obtain the necessary and sufficient condition to be the function f in the class $T(\lambda,\beta,m,n,p,j)$.

Theorem (1): Let $f \in T(p,j)$. Then the function $f \in T(\lambda,\beta,m,n,p,j)$ if and only if

$$\sum_{k=j+p}^{\infty} k\left(\frac{k}{p}\right)^n (k-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right) a_k \le p\beta, \tag{9}$$

where $0 \le \lambda \le 1, 0 < \beta \le 1, (m,n \in \mathbb{N}_0), (m,p,j \in \mathbb{N})$ and $z \in U$.
The result is sharp for the function

$$f(z) = z^p + \frac{p\beta}{k\left(\frac{k}{p}\right)^n (k-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right)} z^k, \quad (k \ge j+p). \tag{10}$$

Proof: Suppose that the inequality (9) holds true and $|z| = 1$.
Then, we have

$$\left|(1-\lambda)z\left(D_p^n f(z)\right)'' + \lambda z\left(D_p^{n+m} f(z)\right)'' - (p-1)(1-\lambda)\left(D_p^n f(z)\right)' - \lambda(p-1)\left(D_p^{n+m} f(z)\right)'\right|$$
$$- \beta\left|(1-\lambda)\left(D_p^n f(z)\right)' + \lambda\left(D_p^{n+m} f(z)\right)'\right|$$
$$= \left|\sum_{k=j+p}^{\infty} k(k-p)\left(\frac{k}{p}\right)^n \left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right) a_k z^{k-p}\right| - \beta\left|p + \sum_{k=j+p}^{\infty} k\left(\frac{k}{p}\right)^n \left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right) a_k z^{k-p}\right|$$
$$\le \sum_{k=j+p}^{\infty} k(k-p)\left(\frac{k}{p}\right)^n \left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right) a_k |z|^{k-p} - p\beta - \beta \sum_{k=j+p}^{\infty} k\left(\frac{k}{p}\right)^n \left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right) a_k |z|^{k-p}$$
$$= \sum_{k=j+p}^{\infty} k\left(\frac{k}{p}\right)^n (k-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right) a_k - p\beta \le 0,$$

by hypothesis.
Hence, by maximum modulus principle, $f \in T(\lambda,\beta,m,n,p,j)$.
Conversely, Suppose that $f \in T(\lambda,\beta,m,n,p,j)$. Then from (8), we have

$$\left| \frac{(1-\lambda)z\left(D_p^n f(z)\right)'' + \lambda z\left(D_p^{n+m} f(z)\right)''}{(1-\lambda)\left(D_p^n f(z)\right)' + \lambda\left(D_p^{n+m} f(z)\right)'} - (p-1) \right|$$
$$= \left| \frac{(1-\lambda)z\left(D_p^n f(z)\right)'' + \lambda z\left(D_p^{n+m} f(z)\right)'' - (p-1)\left((1-\lambda)\left(D_p^n f(z)\right)' + \lambda\left(D_p^{n+m} f(z)\right)'\right)}{(1-\lambda)\left(D_p^n f(z)\right)' + \lambda\left(D_p^{n+m} f(z)\right)'} \right|$$

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$$= \left| \frac{\sum_{k=j+p}^{\infty} k(k-p)\left(\frac{k}{p}\right)^n \left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right) a_k z^{k-p}}{p + \sum_{k=j+p}^{\infty} k\left(\frac{k}{p}\right)^n \left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right) a_k z^{k-p}} \right| < \beta.$$

Since $Re(z) \le |z|$ for all z, we have

$$Re\left\{ \frac{\sum_{k=j+p}^{\infty} k(k-p)\left(\frac{k}{p}\right)^n \left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right) a_k z^{k-p}}{p + \sum_{k=j+p}^{\infty} k\left(\frac{k}{p}\right)^n \left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right) a_k z^{k-p}} \right\} < \beta .$$

If we choose z on the real axis and let $z \to 1^-$, then

$$\sum_{k=j+p}^{\infty} k\left(\frac{k}{p}\right)^n (k-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right) a_k \le p\beta.$$

Finally, sharpness follows if we take

$$f(z) = z^p + \frac{p\beta}{k\left(\frac{k}{p}\right)^n (k-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right)} z^k, \quad (k \ge j+p;\ \ p,j \in \mathbb{N}).$$

Corollary (1): Let $f \in T(\lambda,\beta,m,n,p,j)$. Then

$$a_k \le \frac{p\beta}{k\left(\frac{k}{p}\right)^n (k-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right)}, \quad (k \ge j+p;\ \ p,j \in \mathbb{N}). \tag{11}$$

III. Radii of close-to-convexity, star likeness and convexity.

Using the inequalities (5), (6), (7) and Theorem (1), we can compute the radii of close-to-convexity, starlikeness and convexity.

Theorem (2): Let a function $f \in T(\lambda,\beta,m,n,p,j)$. Then f is p-valently close- to-convex of order ρ $(0 \le \rho < p)$ in the disk $|z| < r_1$, where

$$r_1(\lambda,\beta,m,n,p,j,\rho) = \inf_k \left\{ \frac{(p-\rho)\left(\frac{k}{p}\right)^n (k-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right)}{p\beta} \right\}^{\frac{1}{k-p}}, \quad (k \ge j+p;\ \ p,j \in \mathbb{N}).$$

The result is sharp, with the extermal function f given by (10).

Proof: It is sufficient to show that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \le p - \rho \quad (0 \le \rho < p),$$

for $|z| < r_1\,(\lambda,\beta,m,n,p,j,\rho)$, we have that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \le \sum_{k=j+p}^{\infty} k a_k |z|^{k-p}$$

Thus

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \le p - \rho,$$

if

$$\sum_{k=j+p}^{\infty} \frac{k a_k |z|^{k-p}}{p-\rho} \le 1. \tag{12}$$

Hence, by Theorem (1), (12) will be true if

$$\frac{1}{(p-\rho)} |z|^{k-p} \le \frac{\left(\frac{k}{p}\right)^n (k-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right)}{p\beta},$$

and hence

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$$|z| \leq \left\{ \frac{(p-\rho)\left(\frac{k}{p}\right)^n (k-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right)}{p\beta} \right\}^{\frac{1}{k-p}}, \quad (k \geq j+p;\ p,j \in \mathbb{N}).$$

Setting $|z| = r_1$, we get the desired result.

Theorem (3): Let $f \in T(\lambda,\beta,m,n,p,j)$. Then f is p-valently starlike of order $\rho\ (0 \leq \rho < p)$ in the disk $|z| < r_2$, where

$$r_2(\lambda,\beta,m,n,p,j,\rho) = \inf_k \left\{ \frac{k(p-\rho)\left(\frac{k}{p}\right)^n (k-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right)}{p\beta\,(k-\rho)} \right\}^{\frac{1}{k-p}}, \quad (k \geq j+p;\ p,\mathrm{j} \in \mathbb{N}).$$

The result is sharp for the function f given by (10).

Proof: It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq p-\rho \quad (0 \leq \rho < p),$$

for $|z| < r_2(\lambda,\beta,m,n,p,j,\rho)$, we have

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq \frac{\sum_{k=j+p}^{\infty}(k-p)a_k|z|^{k-p}}{1-\sum_{k=j+p}^{\infty}a_k|z|^{k-p}}.$$

Thus

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq p-\rho,$$

if

$$\sum_{k=j+p}^{\infty}\frac{(k-\rho)}{(p-\rho)}a_k|z|^{k-p} \leq 1. \tag{13}$$

Hence, by Theorem (1), (13) will be true if

$$\frac{(k-\rho)}{(p-\rho)}|z|^{k-p} \leq \frac{k\left(\frac{k}{p}\right)^n (k-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right)}{p\beta},$$

and hence

$$|z| \leq \left\{ \frac{k(p-\rho)\left(\frac{k}{p}\right)^n (k-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right)}{p\beta\,(k-\rho)} \right\}^{\frac{1}{k-p}}, \quad (k \geq j+p;\ p,\mathrm{j} \in \mathbb{N}).$$

Setting $|z| = r_2$, we get the desired result.

Theorem (4): Let $f \in T(\lambda,\beta,m,n,p,j)$. Then f is p-valently convex of order $\rho\ (0 \leq \rho < p)$ in the disk $|z| < r_3$, where

$$r_3(\lambda,\beta,m,n,p,j,\rho) = \inf_k \left\{ \frac{(p-\rho)\left(\frac{k}{p}\right)^n (k-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right)}{\beta\,(k-\rho)} \right\}^{\frac{1}{k-p}}, \quad (k \geq j+p;\ p,\mathrm{j} \in \mathbb{N}).$$

The result is sharp with the external function f given by (10).

Proof: it is sufficient to show that

$$\left|1+\frac{zf''(z)}{f'(z)} - p\right| \leq p-\rho \quad (0 \leq \rho < p),$$

for $|z| < r_3(\lambda,\beta,m,n,p,j,\rho)$, we have

$$\left|1+\frac{zf''(z)}{f'(z)} - p\right| \leq \frac{\sum_{k=j+p}^{\infty}k(k-p)a_k|z|^{k-p}}{p-\sum_{k=j+p}^{\infty}k\,a_k|z|^{k-p}}.$$

Thus

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$$\left|1+\frac{zf''(z)}{f'(z)}-p\right|\leq p-\rho,$$

if

$$\sum_{k=j+p}^{\infty}\frac{k(k-\rho)}{p(p-\rho)}a_k|z|^{k-p}\leq 1. \qquad (14)$$

Hence, by Theorem (1), (14) will be true if

$$\frac{(k-\rho)}{(p-\rho)}|z|^{k-p}\leq\frac{\left(\frac{k}{p}\right)^n(k-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m-\lambda\right)}{\beta},$$

and hence

$$|z|\leq\left\{\frac{(p-\rho)\left(\frac{k}{p}\right)^n(k-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m-\lambda\right)}{\beta\,(k-\rho)}\right\}^{\frac{1}{k-p}},\quad(k\geq j+p;\ p,j\in\mathbb{N}).$$

Setting $|z|=r_3$, we get the desired result.

IV. INTEGRAL MEANS

Definition (2)[7]: Let f,g be analytic in U. Then f is said to be subordinate to g, written $f\prec g$, if there exists a schwarz function $w(z)$, which is analytic in U, with $w(0)=0$ and $|w(z)|<1\,(z\in U)$ such that $f(z)=g\big(w(z)\big),(z\in U)$. In particular, if the function g is univalent in U we have the following:
$f(z)\prec g(z)(z\in U)$if and only if $f(0)=g(0)$ and $f(U)\subset g(U)$.
In1925, Littlewood [6] proved the following subordination result which will be required in our present investigation.
Lemma(1)[6]: If f and g are analytic in U with $f\prec g$, then

$$\int_0^{2\pi}\left|f(re^{i\theta})\right|^{\mu}d\theta\leq\int_0^{2\pi}\left|g(re^{i\theta})\right|^{\mu}d\theta, \qquad (15)$$

where $\mu>0$, $z=re^{i\theta}$ *and* $(0<r<1)$.
Applying Theorem (1) and Lemma (1), we prove the following:
Theorem (5): Let $\mu>0$. If $f\in T(\lambda,\beta,m,n,p,j)$ and suppose that f_s is defined by

$$f_s(z)=z^p+\frac{p\beta}{s\left(\frac{s}{p}\right)^n(s-p-\beta)\left(1+\lambda\left(\frac{s}{p}\right)^m-\lambda\right)}z^s,(s\geq j+p;\ p,j\in\mathbb{N}).$$

If there exists an analytic function w defined by

$$\big(w(z)\big)^{s-p}=\frac{s\left(\frac{s}{p}\right)^n(s-p-\beta)\left(1+\lambda\left(\frac{s}{p}\right)^m-\lambda\right)}{p\beta}\sum_{k=j+p}^{\infty}a_k z^{k-p}.$$

Then, for $z=re^{i\theta}$ and $(0<r<1)$,

$$\int_0^{2\pi}|f(z)|^{\mu}d\theta\leq\int_0^{2\pi}|f_s(z)|^{\mu}d\theta,\quad(\mu>0). \qquad (16)$$

Proof: We must show that

$$\int_0^{2\pi}\left|1+\sum_{k=j+p}^{\infty}a_k z^{k-p}\right|^{\mu}d\theta\leq\int_0^{2\pi}\left|1+\frac{p\beta}{s\left(\frac{s}{p}\right)^n(s-p-\beta)\left(1+\lambda\left(\frac{s}{p}\right)^m-\lambda\right)}z^{s-p}\right|^{\mu}d\theta.$$

By applying Lemma (1), it suffices to show that

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$$1+\sum_{k=j+p}^{\infty} a_k z^{k-p} \prec 1+\frac{p\beta}{s\left(\frac{s}{p}\right)^n (s-p-\beta)\left(1+\lambda\left(\frac{s}{p}\right)^m-\lambda\right)} z^{s-p}.$$

Set

$$1+\sum_{k=j+p}^{\infty} a_k z^{k-p} = 1+\frac{p\beta}{s\left(\frac{s}{p}\right)^n (s-p-\beta)\left(1+\lambda\left(\frac{s}{p}\right)^m-\lambda\right)} (w(z))^{s-p}.$$

We find that

$$(w(z))^{s-p} = \frac{s\left(\frac{s}{p}\right)^n (s-p-\beta)\left(1+\lambda\left(\frac{s}{p}\right)^m-\lambda\right)}{p\beta}\sum_{k=j+p}^{\infty} a_k z^{k-p},$$

which readily yield $w(0)=0$.
Furthermore using (9), we obtain

$$|w(z)|^{s-p} = \left|\frac{s\left(\frac{s}{p}\right)^n (s-p-\beta)\left(1+\lambda\left(\frac{s}{p}\right)^m-\lambda\right)}{p\beta}\sum_{k=j+p}^{\infty} a_k z^{k-p}\right|$$

$$\le |z|^j \left|\sum_{k=j+p}^{\infty} \frac{k\left(\frac{k}{p}\right)^n (k-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m-\lambda\right)}{p\beta} a_k\right| \le |z| < 1.$$

Next, the proof for the first derivative.
Theorem (6): Let $\mu > 0$. If $f \in T(\lambda,\beta,m,n,p,j)$ and

$$f_s(z) = z^p + \frac{p\beta}{s\left(\frac{s}{p}\right)^n (s-p-\beta)\left(1+\lambda\left(\frac{s}{p}\right)^m-\lambda\right)} z^s, \quad (s \ge j+p; p,j \in \mathbb{N}).$$

Then for $z = re^{i\theta}$ and $(0<r<1)$,

$$\int_0^{2\pi} |f'(z)|^\mu d\theta \le \int_0^{2\pi} |f_s'(z)|^\mu d\theta, \qquad (\mu>0). (17)$$

Proof: It is sufficient to show that

$$1+\sum_{k=j+p}^{\infty} \frac{k}{p} a_k z^{k-p} \prec 1+\frac{\beta}{\left(\frac{s}{p}\right)^n (s-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m-\lambda\right)} z^{s-p}.$$

This follows because

$$|w(z)|^{s-p} = \left|\frac{\left(\frac{s}{p}\right)^n (s-p-\beta)\left(1+\lambda\left(\frac{s}{p}\right)^m-\lambda\right)}{\beta}\sum_{k=j+p}^{\infty} \frac{k}{p} a_k z^{k-p}\right|$$

$$\le |z|^j \left|\sum_{k=j+p}^{\infty} \frac{k\left(\frac{k}{p}\right)^n (k-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m-\lambda\right)}{p\beta} a_k\right| \le |z| < 1.$$

The following theorem discuss the subordination condition to $f * g$
Theorem (7): Let g of the form (3) and $f \in T(\lambda,\beta,m,n,p,j)$ be of the form (2) and let for some $s \in \mathbb{N}$,

$$\frac{Q_s}{b_s} = \min_{k \ge j+p} \frac{Q_k}{b_k},$$

where

$$Q_k = \frac{k\left(\frac{k}{p}\right)^n (k-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m-\lambda\right)}{p\beta}.$$

Also, let for such $s \in \mathbb{N}$, the functions f_s and g_s be defined respectively by

$$f_s(z) = z^p + \frac{p\beta}{s\left(\frac{s}{p}\right)^n (s-p-\beta)\left(1+\lambda\left(\frac{s}{p}\right)^m - \lambda\right)} z^s,$$

$$g_s(z) = z^p + b_s z^s. \tag{18}$$

If there exists an analytic function w defined by

$$\left(w(z)\right)^{s-p} = \frac{s\left(\frac{s}{p}\right)^n (s-p-\beta)\left(1+\lambda\left(\frac{s}{p}\right)^m - \lambda\right)}{p\beta b_s} \sum_{k=j+p}^{\infty} a_k b_k z^{k-p},$$

then , for $\mu > 0$ and $z = re^{i\theta}$ and $(0 < r < 1)$,

$$\int_0^{2\pi} |(f*g)(z)|^\mu d\theta \le \int_0^{2\pi} |(f_s*g_s)(z)|^\mu d\theta, \qquad (\mu > 0).$$

Proof: convolution of f and g is defined as:

$$(f*g)(z) = z^p + \sum_{k=j+p}^{\infty} a_k b_k.$$

Similarly, from (18), we obtain

$$(f_s*g_s)(z) = z^p + \frac{p\beta b_s}{s\left(\frac{s}{p}\right)^n (s-p-\beta)\left(1+\lambda\left(\frac{s}{p}\right)^m - \lambda\right)} z^s.$$

To prove the theorem, we must show that for $\mu > 0$ and $z = re^{i\theta}$ and $(0 < r < 1)$,

$$\int_0^{2\pi} \left|1 + \sum_{k=j+p}^{\infty} a_k b_k z^{k-p}\right|^\mu d\theta \le \int_0^{2\pi} \left|1 + \frac{p\beta b_s}{s\left(\frac{s}{p}\right)^n (s-p-\beta)\left(1+\lambda\left(\frac{s}{p}\right)^m - \lambda\right)} z^{s-p}\right|^\mu d\theta.$$

Thus, by applying Lemma (1), it would suffice to show that

$$1 + \sum_{k=j+p}^{\infty} a_k b_k z^{k-p} \prec 1 + \frac{p\beta b_s}{s\left(\frac{s}{p}\right)^n (s-p-\beta)\left(1+\lambda\left(\frac{s}{p}\right)^m - \lambda\right)} z^{s-p}. \tag{19}$$

If the subordination (19) true, then there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ such that

$$1 + \sum_{k=j+p}^{\infty} a_k b_k z^{k-p} = 1 + \frac{p\beta b_s}{s\left(\frac{s}{p}\right)^n (s-p-\beta)\left(1+\lambda\left(\frac{s}{p}\right)^m - \lambda\right)} (w(z))^{s-p}.$$

From the hypothesis of the theorem, there exists an analytic function w given by

$$(w(z))^{s-p} = \frac{s\left(\frac{s}{p}\right)^n (s-p-\beta)\left(1+\lambda\left(\frac{s}{p}\right)^m - \lambda\right)}{p\beta b_s} \sum_{k=j+p}^{\infty} a_k b_k z^{k-p},$$

which readily yield $w(0) = 0$. Thus for such function w, using the hypothesis in the coefficient inequality for the class $T(\lambda, \beta, m, n, p, j)$, we get

$$|w(z)|^{s-p} = \left|\frac{s\left(\frac{s}{p}\right)^n (s-p-\beta)\left(1+\lambda\left(\frac{s}{p}\right)^m - \lambda\right)}{p\beta b_s} \sum_{k=j+p}^{\infty} a_k b_k z^{k-p}\right|$$

$$\le |z|^j \left|\frac{s\left(\frac{s}{p}\right)^n (s-p-\beta)\left(1+\lambda\left(\frac{s}{p}\right)^m - \lambda\right)}{p\beta b_s} \sum_{k=j+p}^{\infty} a_k b_k\right| \le |z| < 1$$

Therefore, the subordination (19) holds true.

Now , we discuss the integral means inequalities for $f \in T(\lambda, \beta, m, n, p, j)$ and h defined by

$$h(z) = z^p + b_s z^s + b_{2s-p} z^{2s-p}, \qquad (b_s \ge 0, s \ge j+p). \tag{20}$$

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Theorem (8): Let $f \in T(\lambda, \beta, m, n, p, j)$ and h given by (20).
If f satisfies

$$\sum_{k=j+p}^{\infty} a_k \leq b_{2s-p} - b_s, \quad (b_s < b_{2s-p}), \tag{21}$$

and there exists an analytic function w such that

$$b_{2s-p}\big(w(z)\big)^{2(s-p)} + b_s\big(w(z)\big)^{s-p} - \sum_{k=j+p}^{\infty} a_k z^{k-p} = 0.$$

Then, for $z = re^{i\theta}$ and $(0 < r < 1)$,

$$\int_0^{2\pi} |f(z)|^{\mu}\, d\theta \leq \int_0^{2\pi} |h(z)|^{\mu}\, d\theta, \qquad (\mu > 0).$$

Proof: By putting $z = re^{i\theta}\,(0 < r < 1)$, we see that

$$\int_0^{2\pi} |f(z)|^{\mu}\, d\theta = \int_0^{2\pi} \left| z^p + \sum_{k=j+p}^{\infty} a_k z^k \right|^{\mu} d\theta = r^{p\mu} \int_0^{2\pi} \left| 1 + \sum_{k=j+p}^{\infty} a_k z^{k-p} \right| d\theta$$

and

$$\int_0^{2\pi} |h(z)|^{\mu}\, d\theta = \int_0^{2\pi} \left| z^p + b_s z^s + b_{2s-p} z^{2s-p} \right|^{\mu} d\theta \; = r^{p\mu} \int_0^{2\pi} \left| 1 + b_s z^{s-p} + b_{2s-p} z^{2(s-p)} \right|^{\mu} d\theta.$$

Applying Lemma (1), we have to show that

$$1 + \sum_{k=j+p}^{\infty} a_k z^{k-p} \prec 1 + b_s z^{s-p} + b_{2s-p} z^{2(s-p)}.$$

Let us define the function w by

$$1 + \sum_{k=j+p}^{\infty} a_k z^{k-p} = 1 + b_s\big(w(z)\big)^{s-p} + b_{2s-p}\big(w(z)\big)^{2(s-p)},$$

or by

$$b_{2s-p}\big(w(z)\big)^{2(s-p)} + b_s\big(w(z)\big)^{s-p} - \sum_{k=j+p}^{\infty} a_k z^{k-p} = 0. \tag{22}$$

Since for $z = 0$, $\big(w(0)\big)^{s-p}\{b_{2s-p}\big(w(0)\big)^{s-p} + b_s\} = 0$.
There exists an analytic function w in U such that $w(0) = 0$.
Next, we prove the analytic function w satisfies $|w(z)| < 1, (z \in U)$ for the condition (21). By (22), we know that,

$$\left| b_{2s-p}\big(w(z)\big)^{2(s-p)} + b_s\big(w(z)\big)^{s-p} \right| = \left| \sum_{k=j+p}^{\infty} a_k z^{k-p} \right| < \sum_{k=j+p}^{\infty} a_k.$$

For $z \in U$, hence

$$b_{2s-p}|w(z)|^{2(s-p)} - b_s|w(z)|^{s-p} - \sum_{k=j+p}^{\infty} a_k < 0. \tag{23}$$

Letting $t = |w(z)|^{s-p}\,(t \geq 0)$ in (23), we define the function $G(t)$ by

$$G(t) = b_{2s-p} t^2 - b_s t - \sum_{k=j+p}^{\infty} a_k.$$

If $G(1) \geq 0$, then we have $t < 1$ for $G(t) < 0$. Indeed we have

$$G(1) = b_{2s-p} - b_s - \sum_{k=j+p}^{\infty} a_k \geq 0.$$

That is $\sum_{k=j+p}^{\infty} a_k \leq b_{2s-p} - b_s$.

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V. Classes of Preserving Integral Operators

Theorem (9): Let $f(z) \in T(\lambda,\beta,m,n,p,j)$ be defined by (2) and c be any real number such that $c > -p$, then the integral operator

$$G(z) = \frac{c+p}{z^c}\int_0^z s^{c-1} f(s)ds, \qquad c > -p \tag{24}$$

also belongs to $T(\lambda,\beta,m,n,p,j)$.

Proof: By virtue of (24) it follows from (2) that

$$G(z) = \frac{c+p}{z^c}\int_0^z s^{c-1}\left(s^p + \sum_{k=j+p}^{\infty} a_k\, s^k\right)ds \quad = \frac{c+p}{z^c}\int_0^z \left(s^{p+c-1} + \sum_{k=j+p}^{\infty} a_k\, s^{k+c-1}\right)ds$$

$$= z^p + \sum_{k=j+p}^{\infty}\left(\frac{c+p}{c+k}\right)a_k\, z^k = z^p + \sum_{k=j+p}^{\infty} h_k\, z^k \ where\ h_k = \left(\frac{c+p}{c+k}\right)a_k.$$

$$= z^p + \sum_{k=j+p}^{\infty} h_k\, z^k \ where\ h_k = \left(\frac{c+p}{c+k}\right)a_k.$$

But

$$\sum_{k=j+p}^{\infty} k\left(\frac{k}{p}\right)^n (k-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right)h_k = \sum_{k=j+p}^{\infty} k\left(\frac{k}{p}\right)^n (k-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right)\left(\frac{c+p}{c+k}\right)a_k.$$

Since $\left(\frac{c+p}{c+k}\right) \le 1$ and by (9) the last expression is less than or equal $p\beta$, so the proof is complete.

Theorem (10): Let $c \in \mathbb{R}\ (c > -p)$ if $G(z) \in T(\lambda,\beta,m,n,p,j)$, then the function $f(z)$ defined by (24), is p-valent in $|z| < r_4$ where

$$r_4 = \inf_{k \geq j+p}\left\{\frac{(c+p)\left(\frac{k}{p}\right)^n (k-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right)}{\beta\,(c+k)}\right\}^{\frac{1}{k-p}}, \tag{25}$$

the result is sharp.

Proof: Let $G(z) = z^p + \sum_{k=j+p}^{\infty} h_k\, z^k = \frac{c+p}{z^c}\int_0^z s^{c-1} f(s)ds$ so

$$f(z) = z^p + \sum_{k=j+p}^{\infty}\left(\frac{c+k}{c+p}\right)h_k\, z^k, \qquad c > -p.$$

Thus it is enough to show that $\left|\frac{f'(z)}{z^{p-1}} - p\right| \le p,\ |z| < r_4$. But

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| = \left|\sum_{k=j+p}^{\infty} k\left(\frac{c+k}{c+p}\right)h_k\, z^{k-p}\right|,$$

then

$$\sum_{k=j+p}^{\infty}\frac{k}{p}\left(\frac{c+k}{c+p}\right)h_k\,|z|^{k-p} \le 1. \tag{26}$$

Since $G(z) \in T(\lambda,\beta,m,n,p,j)$ by (9) we have

$$\sum_{k=j+p}^{\infty}\frac{k\left(\frac{k}{p}\right)^n (k-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right)}{p\beta}h_k \le 1.$$

Therefore (26) will be true if

$$\left(\frac{c+k}{c+p}\right)|z|^{k-p} \le \frac{\left(\frac{k}{p}\right)^n (k-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right)}{\beta}$$

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or

$$|z| \leq \left\{ \frac{(c+p)\left(\frac{k}{p}\right)^n (k-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right)}{\beta(c+k)} \right\}^{\frac{1}{k-p}}, (k \geq j+p; p, j \in \mathbb{N})$$

and this proves the result. Sharpness of this theorem follows if we put

$$f(z) = z^p + \frac{p\beta(c+k)}{k\left(\frac{k}{p}\right)^n (k-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right)(c+p)} z^k, \qquad (k \geq j+p; p, j \in \mathbb{N}). \tag{27}$$

Theorem (12): Let $f(z) \in T(\lambda, \beta, m, n, p, j)$, then the integral operator

$$F_\gamma(z) = (1-\gamma)z^p + \gamma p \int_0^z \frac{f(s)}{s} ds \quad (\gamma \geq 0, \ z \in U), \tag{28}$$

is also in $T(\lambda, \beta, m, n, p, j)$ if $0 \leq \gamma \leq \frac{j+p}{p}$.

Proof: If $f(z) = z^p + \sum_{k=j+p}^{\infty} a_k z^k$, then

$$F_\gamma(z) = (1-\gamma)z^p + \gamma p \int_0^z \left(\frac{s^p + \sum_{k=j+p}^{\infty} a_k s^k}{s} \right) ds = (1-\gamma)z^p + \gamma p \left(\frac{1}{p} z^p + \sum_{k=j+p}^{\infty} \frac{a_k}{k} z^k \right)$$

$$= z^p + \sum_{k=j+p}^{\infty} \frac{\gamma p}{k} a_k z^k = z^p + \sum_{k=j+p}^{\infty} g_k z^k,$$

where $g_k = \frac{\gamma p}{k} a_k$. But

$$\sum_{k=j+p}^{\infty} k\left(\frac{k}{p}\right)^n (k-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right) g_k = \sum_{k=j+p}^{\infty} k\left(\frac{k}{p}\right)^n (k-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right) \frac{\gamma p}{k} a_k$$

$$\leq \sum_{k=j+p}^{\infty} k\left(\frac{k}{p}\right)^n (k-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right) \frac{\gamma p}{j+p} a_k$$

$$\leq \sum_{k=j+p}^{\infty} k\left(\frac{k}{p}\right)^n (k-p-\beta)\left(1+\lambda\left(\frac{k}{p}\right)^m - \lambda\right) a_k \left(since\ \frac{\gamma p}{j+p} \leq 1 \right) \leq p\beta.$$

So the proof is complete.

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