ISSN: 2350-0328

International Journal of AdvancedResearch in Science, Engineering and Technology

Vol. 5, Issue 8 , August 2018

Weak forms of Mixed Continuity

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ABSTRACT: Multivalued functions between topological spaces have applications to the fixed point theory which in turn has applications to social science, science and engineering. The authors have recently studied upper mixed continuous and lower mixed continuous multifunctions between topological spaces. The purpose of this paper is to introduce and characterize some weak forms of upper and lower mixed continuous multifunctions.

KEYWORDS: Multifunctions, upper mixed continuity; lower mixed continuity.

MSC(2010): 54C60

I. INTRODUCTION AND PRELIMINARIES

Functions between two sets play a significant role in mathematics. In particular multivalued functions have applications in applied mathematics such as optimal control, calculus of variations, probability, statistics, differential inclusions, fixed point theory and so on.

Let X and Y be any two non empty sets. A multivalued function from X to Y, is a function $F:X\to \wp(Y)$ such that $F(x)\neq\emptyset$ for every $x\in X$ where $\wp(Y)$ denotes the power set of Y. A multifunction F from X to Y is denoted by $F:X\twoheadrightarrow Y$. If $F:X\twoheadrightarrow Y$ then for every subset A of Y, $F^+(A)=\{x\in X:F(x)\subseteq A\}$ is called the upper inverse of A and $F^-(A)=\{x\in X:F(x)\cap A\neq\emptyset\}$is called the lower inverse of A. For every subset A of X, $F(A)=\bigcup_{x\in A} F(x)$. For the basic results in multivalued analysis one may consult [2, 7, 9, 12]. The applications of multivalued functions are found in [6, 8, 10, 11, 16, 17, 18, 22, 23, 24, 29]. The properties of multifunctions are discussed in [19, 22, 25]. The continuity of multifunctions is studied in [5, 10,13,14,20,21, 23, 26,28, 31-38]. It is easy to see that $F^+(A)\subseteq F^-(A)$ for every subset A of X. The reverse inclusion is not true. Throughout this paper $F:X\twoheadrightarrow Y$ is a multi valued function, A , B are the subsets of X and 'iff' denotes ' if and only if '. The following lemmas will be useful to study the continuity of multifunctions.

Lemma 1.1:

(i) $F(A\cup B) = F(A)\cup F(B)$ but $F(A\cap B)\subseteq F(A)\cap F(B)$

(ii) $F(X\setminus A) \supseteq F(X)\setminus F(A)$

(iii) $A\subseteq B\Rightarrow F(A)\subseteq F(B)$

Lemma 1.2:

(i) $F^+(Y\setminus A) = X\setminus F^-(A)$ and $F^-(Y\setminus A) = X \setminus F^+(A)$

(ii) $A\subseteq B\Rightarrow F^+(A)\subseteq F^+(B)$ and $F^-(A)\subseteq F^-(B)$

(iii) $F^-(A\cup B) = F^-(A) \cup F^-(B)$ but $F^-(A\cap B) \supseteq F^-(A) \cap F^-(B)$

(iv) $F^+(A\cup B) \subseteq F^+(A) \cup F^+(B)$ but $F^+(A\cap B) = F^+(A) \cap F^+(B)$

(v) If $u\in F^+(F(x))$ then $F(u) \subseteq F(x)$

(vi) $F(F^+(A)) \subseteq A\subseteq F(F^-(A))$

Definition 1.3: Let $F:X\twoheadrightarrow Y$ and $G:X\twoheadrightarrow Y$. The multifunctions $F\cup G$ and $F\cap G$ are defined as $(F\cup G)(x) = F(x)\cup G(x)$ and $(F\cap G)(x) = F(x)\cap G(x)$ for every $x\in X$.

Definition 1.4: Let $F:X\twoheadrightarrow Y$ and $G:X\twoheadrightarrow Z$. The multifunction $F\times G:X\twoheadrightarrow Y\times Z$ is defined as $(F\times G)(x) = F(x)\times G(x)$ for every $x\in X$.

ISSN: 2350-0328
International Journal of AdvancedResearch in Science, Engineering and Technology
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Definition 1.5: Let $F:X \twoheadrightarrow Y$ and $G:Y \twoheadrightarrow Z$. The multifunction $G°F :X \twoheadrightarrow Z$ is defined as $(G°F)(x)= \bigcup_{y \in F(x)} G(y)$ for every $x \in X$.

Lemma 1.6: Let $F:X \twoheadrightarrow Y$ and $G:X \twoheadrightarrow Y$. Let $V \subseteq Y$. Then

(i) $(F\cup G)^{-}(V) = F^{-}(V) \cup G^{-}(V)$ but $(F\cap G)^{-}(V) \subseteq F^{-}(V) \cap G^{-}(V)$

(ii) $(F\cup G)^{+}(V) \subseteq F^{+}(V) \cup G^{+}(V)$ and $(F\cap G)^{+}(V) \subseteq F^{+}(V) \cap G^{+}(V)$

The concepts of regular open[27], α -open[18], semiopen[19], preopen[15], b-open[4] and β-open[1] sets were introduced and studied by Stone, Njastad, Levine, Mashhour et.al., Andrijevic and Abdel Monsef et.al. respectively . The β-open sets are also called semi-preopen sets in the sense of Andrijevic[3]. The notations RO(X,τ), α O(X,τ), SO(X,τ), PO(X,τ), bO(X,τ)and βO(X,τ) denote the collection of regular open sets, α -open sets, semiopen sets, preopen sets, b-open sets and β-open sets in a topological space (X,τ) respectively. Similarly the notations RC(X,τ), αC(X,τ), SC(X,τ), PC(X,τ), bC(X,τ)and βC(X,τ) denote the collection of regular closed sets, α -closed sets, semiclosed sets, preclosed sets, b-closed sets and β-closed sets in a topological space (X,τ) respectively.

Definition 1.7: $F: (X, \tau) \twoheadrightarrow (Y, \sigma)$ is

(i) upper continuous (briefly U.C) if for all $x \in X$ and for every $V \in \sigma$ with $F(x) \subseteq V$ there exists $U \in \tau$ containing x such that $F(U) \subseteq V$.

(ii) upper weakly continuous (briefly U.WC) if for all $x \in X$ and for every $V \in \sigma$ with $F(x) \subseteq V$ there exists an open set U in (X, τ) containing x such that $F(U) \subseteq Cl\, V$.

(iii) upper α-continuous (briefly U. αC) if for all $x \in X$ and for every $V \in \sigma$ with $F(x) \subseteq V$ there exists an α-open set U in (X,τ) containing x such that $F(U) \subseteq V$.

(iv) upper precontinuous (briefly U.preC) if for all $x \in X$ and for every $V \in \sigma$ with $F(x) \subseteq V$ there exists a preopen set U in (X, τ) containing x such that $F(U) \subseteq V$.

(v) upper b-continuous (briefly U.bC) if for all $x \in X$ and for every $V \in \sigma$ with $F(x) \subseteq V$ there exists a b-open set U in (X, τ) containing x such that $F(U) \subseteq V$.

(vi) upper β-continuous (briefly U.βC) if for all $x \in X$ and for every $V \in \sigma$ with $F(x) \subseteq V$ there exists an β-open set U in (X, τ) containing x such that $F(U) \subseteq V$.

Definition 1.8: $F: (X, \tau) \twoheadrightarrow (Y, \sigma)$ is

(i) lower continuous (briefly L.C) if for all $x \in X$ and for every $V \in \sigma$ with $F(x) \cap V \neq \varnothing$ there exists $U \in \tau$ containing x such that $F(u) \cap V \neq \varnothing$ for every $u \in U$.

(ii) lower weakly continuous (briefly L.WC) if for all $x \in X$ and for every $V \in \sigma$ with $F(x) \cap V \neq \varnothing$ there exists an open set U in (X, τ) containing x such that $F(u) \cap Cl V \neq \varnothing$ for every $u \in U$.

(iii) lower α-continuous (briefly L.αC) if for all $x \in X$ and for every $V \in \sigma$ with $F(x) \cap V \neq \varnothing$, there exists an α-open set U in (X,τ) containing x such that $F(u) \cap V \neq \varnothing$ for every $u \in U$.

(iv) lower precontinuous (briefly L.preC) if for all $x \in X$ and for every $V \in \sigma$ with $F(x) \cap V \neq \varnothing$ there exists a preopen set U in (X,τ) containing x such that $F(u) \cap V \neq \varnothing$ for every $u \in U$.

(v) lower b-continuous (briefly L.bC) if for all $x \in X$ and for every $V \in \sigma$ with $F(x) \cap V \neq \varnothing$ there exists a b-open set U in (X, τ) containing x such that $F(u) \cap V \neq \varnothing$ for every $u \in U$.

(vi) lower β-continuous (briefly L. βC) if for all $x \in X$ and for every $V \in \sigma$ with $F(x) \cap V \neq \varnothing$ there exists an β-open set U in (X,τ) containing x such that $F(u) \cap V \neq \varnothing$ for every $u \in U$.

Definition 1.9: $F: (X, \tau) \twoheadrightarrow (Y, \sigma)$ is upper mixed continuous(briefly U.M.C) [30] if for all $x \in X$ and for every $V \in \sigma$ with $F(x) \subseteq V$ there exists $U \in \tau$ containing x such that $F(u) \cap V \neq \varnothing$ for every $u \in U$ and is lower mixed continuous (briefly L.M.C)[30] if for all $x \in X$ and for every $V \in \sigma$ with $F(x) \cap V \neq \varnothing$ there exists $U \in \tau$ containing x such that $F(U) \subseteq V$.

ISSN: 2350-0328

International Journal of AdvancedResearch in Science, Engineering and Technology

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II. Weak forms of upper mixed continuous multifunctions

Definition 2.1: $F:(X,\tau)\twoheadrightarrow(Y,\sigma)$ is

(i)upper mixed weakly continuous (briefly U.M.WC) if for all $x\in X$ and for every $V\in\sigma$ with $F(x)\subseteq V$ there exists $U\in\tau$ containing x such that $F(u)\cap \mathit{Cl}V \neq\varnothing$ for every $u\in U$,

(ii) upper mixed b-continuous (briefly U.M.bC) if for all $x\in X$ and for every $V\in\sigma$ with $F(x)\subseteq V$ there exists $U\in bO(X,\tau)$ containing x such that $F(u)\cap V\neq\varnothing$ for every $u\in U$,

(iii) upper mixed α-continuous (briefly U.M.αC) if for all $x\in X$ and for every $V\in\sigma$ with $F(x)\subseteq V$ there exists $U\in\alpha O(X,\tau)$ containing x such that $F(u)\cap V\neq\varnothing$ for every $u\in U$,

(iv) upper mixed β-continuous (briefly U.M. βC) if for all $x\in X$ and for every $V\in\sigma$ with $F(x)\subseteq V$ there exists $U\in\beta O(X,\tau)$ containing x such that $F(u)\cap V\neq\varnothing$ for every $u\in U$,

(v) upper mixed precontinuous (briefly U.M. preC) if for all $x\in X$ and for every $V\in\sigma$ with $F(x)\subseteq V$ there exists $U\in PO(X,\tau)$ containing x such that $F(u)\cap V\neq\varnothing$ for every $u\in U$,

Since every open set is α-open, β-open, b-open and preopen , it follows from the above definition that if $F:(X,\tau)\to(Y,\sigma)$ is upper mixed continuous then it is upper mixed b-continuous, upper mixed α-continuous. upper mixed β-continuous and upper mixed precontinuous functions. Also every upper mixed continuous function is upper mixed weakly continuous. Hence the Definition 2.1 is meaningful as the functions in the definition are the weaker forms of upper mixed continuous functions.

Proposition 2.2 : $F:(X,\tau)\twoheadrightarrow(Y,\sigma)$ is

(i) U.M.αC iff $F^+(V)\subseteq \alpha\mathit{Int}\,F^-(V)$ for every $V\in\sigma$.

(ii) U.M.preC iff $F^+(V)\subseteq p\mathit{Int}\,F^-(V)$ for every $V\in\sigma$.

(iii) U.M.βC iff $F^+(V)\subseteq \beta\mathit{Int}\,F^-(V)$ for every $V\in\sigma$.

(iv) U.M.bC iff $F^+(V)\subseteq b\mathit{Int}\,F^-(V)$ for every $V\in\sigma$.

(v) U.M.WC iff $F^+(\mathit{Cl}V)\subseteq \mathit{Int}\,F^-(\mathit{Cl}V)$ for every $V\in\sigma$.

Proof: Suppose $F:(X,\tau)\twoheadrightarrow(Y,\sigma)$ is U.M.αC. Let $x\in F^+(V)$ and $V\in\sigma$. Then $F(x)\subseteq V$. Since F is U.M.αC at x, there exists $U\in\alpha O(X,\tau)$ containing x such that $F(u)\cap V\neq\varnothing$ for every $u\in U$. This shows that $u\in F^-(V)$ for every $u\in U$ that implies $x\in U\subseteq F^-(V)$. Since $F^+(V)\subseteq F^-(V)$, the above arguments show that every point of $F^+(V)$ is an α-interior point of $F^-(V)$ that implies $F^+(V)\subseteq \alpha\mathit{Int}\,F^-(V)$.

Conversely let $F^+(V)\subseteq \alpha\mathit{Int}\,F^-(V)$ for every $V\in\sigma$. Suppose $F(x)\subseteq V$ and $V\in\sigma$. Then $x\in F^+(V)\subseteq \alpha\mathit{Int}\,F^-(V)$. This implies that there exists $U\in\alpha O(X,\tau)$ containing x such that $x\in U\subseteq F^-(V)$. This shows that $u\in F^-(V)$ for every $u\in U$ that implies $F(u)\cap V\neq\varnothing$ for every $u\in U$. This proves that F is U.M.αC. The proof for (ii) ,(iii) and (iv) is similar.

Suppose $F:(X,\tau)\twoheadrightarrow(Y,\sigma)$ is U.M.WC. Let $x\in F^+(V)$ and $V\in\sigma$. Then $F(x)\subseteq V$. Since F is U.M.WC at x, there exists $U\in\tau$ containing x such that $F(u)\cap \mathit{Cl}V\neq\varnothing$ for every $u\in U$. This shows that $u\in F^-(\mathit{Cl}V)$ for every $u\in U$ that implies $x\in U\subseteq F^-(\mathit{Cl}V)$. Since $F^+(\mathit{Cl}V)\subseteq F^-(\mathit{Cl}V)$, the above arguments show that every point of $F^+(\mathit{Cl}V)$ is an interior point of $F^-(\mathit{Cl}V)$ that implies $F^+(\mathit{Cl}V)\subseteq \mathit{Int}\,F^-(\mathit{Cl}V)$.

Conversely let $F^+(\mathit{Cl}V)\subseteq\mathit{Int}\,F^-(\mathit{Cl}V)$ for every $V\in\sigma$. Suppose $F(x)\subseteq V$ and $V\in\sigma$. Then $x\in F^+(V)\subseteq F^+(\mathit{Cl}V)\subseteq\mathit{Int}\,F^-(\mathit{Cl}V)$. This implies that there exists $U\in\tau$ containing x such that $x\in U\subseteq F^-(\mathit{Cl}V)$. This shows that $u\in F^-(\mathit{Cl}V)$ for every $u\in U$ that implies $F(u)\cap \mathit{Cl}V\neq\varnothing$ for every $u\in U$. This proves that F is U.M.WC. This proves (v).

Proposition 2.3 : $F:(X,\tau)\to(Y,\sigma)$ is

(i) U.M.αC iff $\alpha\mathit{Cl}F^+(V)\subseteq F^-(V)$ for every closed set V in (Y,σ).

(ii) U.M.preC iff $p\mathit{Cl}F^+(V)\subseteq F^-(V)$ for every closed set V in (Y,σ).

(iii) U.M.βC iff $\beta\mathit{Cl}F^+(V)\subseteq F^-(V)$ for every closed set V in (Y,σ).

(iv) U.M.bC iff $b\mathit{Cl}F^+(V)\subseteq F^-(V)$ for every closed set V in (Y,σ).

(v) U.M.WC iff $\mathit{Cl}\,(F^+(\mathit{Int}B))\subseteq F^-(\mathit{Int}B)$ for every closed set B in (Y,σ).

Proof: F is U.M.αC iff $F^+(Y\backslash B)\subseteq\alpha\mathit{Int}\,F^-(Y\backslash B)$ for every closed set B in (Y,σ).

iff $X\backslash F^-(B)\subseteq\alpha\mathit{Int}\,(X\backslash F^+(B)) = X\backslash\alpha\mathit{Cl}\,(F^+(B))$

iff $\alpha\mathit{Cl}\,(F^+(B))\subseteq F^-(B)$ for every closed set B in (Y,σ).

ISSN: 2350-0328
International Journal of AdvancedResearch in Science, Engineering and Technology
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This proves (i) and the proof for (ii) ,(iii) and (iv) is analog.
F is U.M.WC iff $F^+(Cl\,(Y\backslash B))\subseteq Int\, F^-(\,Cl\,(Y\backslash B))$ for every closed set B in (Y, σ).
iff $F^+((Y\backslash IntB))\subseteq Int\, F^-(\,(Y\backslash IntB))$
iff $X\backslash F^-(IntB)\subseteq Int(X\backslash F^+(IntB))= X\backslash Cl\, F^+(IntB)$
iff $Cl\,(F^+(\,IntB))\subseteq F^-(\,IntB)$ for every closed set B in (Y, σ).
This proves (v).

Proposition 2.4 : If $F:(X, \tau)\twoheadrightarrow (Y , \sigma)$ and $G: (X, \tau)\twoheadrightarrow (Y , \sigma)$ are U.M.αC. then $F\cup G$ is U.M.αC. .
Proof: Suppose $F: (X, \tau)\twoheadrightarrow (Y , \sigma)$ and $G: (X, \tau)\twoheadrightarrow (Y, \sigma)$ are U.M.αC. Let $x\in X$ and $V\in\sigma$ with $(F\cup G)(x)\subseteq V$. Then $F(x)\subseteq V$ and $G(x)\subseteq V$. Since $F:(X, \tau)\to(Y , \sigma)$ and $G:(X, \tau)\to(Y , \sigma)$ are U.M.αC at x, there are α-open sets U_1 and U_2 in X such that $F(x')\cap V \neq\emptyset$ for every $x'\in U_1$ and $G(x'')\cap V \neq\emptyset$ for every $x''\in U_2$. Then $U = U_1\cap U_2$ is the required α-open set satisfying $(F\cup G)(u)\cap V\neq\emptyset$ for $u\in U$.

Proposition 2.5 : If $F:(X, \tau)\twoheadrightarrow (Y , \sigma)$ and $G: (X, \tau)\twoheadrightarrow (Y , \sigma)$ are U.M.WC. then $F\cup G$ is U.M.WC.
Proof: Suppose $F: (X, \tau)\twoheadrightarrow (Y , \sigma)$ and $G: (X, \tau)\twoheadrightarrow (Y , \sigma)$ are U.M.WC. Let $x\in X$ and $V\in\sigma$ with $(F\cup G)(x)\subseteq V$. Then $F(x)\subseteq V$ and $G(x)\subseteq V$. Since $F:(X, \tau)\to(Y , \sigma)$ and $G:(X, \tau)\to(Y , \sigma)$ are U.M.WC at x, there are open sets U_1 and U_2 in X such that $F(x')\cap ClV \neq\emptyset$ for every $x'\in U_1$ and $G(x'')\cap ClV \neq\emptyset$ for every $x''\in U_2$. Then $U = U_1\cap U_2$ is the required open set satisfying $(F\cup G)(u)\cap ClV\neq\emptyset$ for $u\in U$.

Proposition 2.6: Let $G:(Y,\sigma) \twoheadrightarrow (Z , \eta)$ be U.C
(i)If $F: (X, \tau)\twoheadrightarrow (Y,\sigma)$ is U.M.αC and then $G^\circ F: (X, \tau)\twoheadrightarrow (Z , \eta)$ is U.M.αC .
(ii)If $F: (X, \tau)\twoheadrightarrow (Y,\sigma)$ is U.M.preC and then $G^\circ F: (X, \tau)\twoheadrightarrow (Z , \eta)$ is U.M.preC .
(iii)If $F: (X, \tau)\twoheadrightarrow (Y,\sigma)$ is U.M.βC and then $G^\circ F: (X, \tau)\twoheadrightarrow (Z , \eta)$ is U.M.βC .
(iv)If $F: (X, \tau)\twoheadrightarrow (Y,\sigma)$ is U.M.βC and then $G^\circ F: (X, \tau)\twoheadrightarrow (Z , \eta)$ is U.M.βC .

Proof: Suppose $F: (X, \tau)\twoheadrightarrow (Y,\sigma)$ is U.M.αC and $G:(Y,\sigma) \twoheadrightarrow (Z , \eta)$ is U.C. Let $x\in X$ and $V\in\eta$ with $G^\circ F(x)\subseteq V$. Then $G(y) \subseteq V$ for every $y\in F(x)$. That is $y\in G^+(V)$ for every $y\in F(x)$. Since G is U.C, $G^+(V)$ is open in Y. Clearly $F(x)\subseteq G^+(V)$. Since F is U.M.αC, there is an α-open set U in X containing x such that $F(u)\cap G^+(V)\neq\emptyset$ for every $u\in U$. That is $G(F(u)\cap G^+(V))\neq\emptyset$ for every $u\in U$ that implies $G(F(u))\cap G(G^+(V))\neq\emptyset$ for every $u\in U$. This shows that $G(F(u))\cap V\neq\emptyset$ for every $u\in U$ that implies $G^\circ F$ is U.M.αC. The proof for (ii) ,(iii) and (iv) is similar.

Proposition 2.7: Let $G:(Y,\sigma) \twoheadrightarrow (Z , \eta)$ be U.C such that $G(ClV) \subseteq ClG(V)$ for every open set V in (Y,σ). Then if $F: (X, \tau)\twoheadrightarrow (Y,\sigma)$ is U.M.WC and then $G^\circ F: (X, \tau)\twoheadrightarrow (Z , \eta)$ is U.M.WC

Proof: Suppose $F: (X, \tau)\to(Y,\sigma)$ is U.M.WC and $G:(Y,\sigma) \twoheadrightarrow (Z , \eta)$ is U.C. Let $x\in X$ and $V\in\eta$ with $G^\circ F(x)\subseteq V$. Then $G(y) \subseteq V$ for every $y\in F(x)$. That is $y\in G^+(V)$ for every $y\in F(x)$. Since G is U.C, $G^+(V)$ is open in Y. Clearly $F(x)\subseteq G^+(V)$. Since F is U.M.WC, there is an open set U in X containing x such that $F(u)\cap Cl(G^+(V))\neq\emptyset$ for every $u\in U$. That is $G(F(u)\cap Cl(G^+(V)))\neq\emptyset$ for every $u\in U$ that implies $G(F(u))\cap G(ClG^+(V)))\neq\emptyset$ for every $u\in U$.
Since $G(ClV) \subseteq ClG(V)$ for every open set V in (Y,σ) and since $G^+(V)$ is open in (Y,σ) we have $G(F(u))\cap Cl(G(G^+(V)))\neq\emptyset$ that implies $G(F(u))\cap V\neq\emptyset$ for every $u\in U$ This implies $G^\circ F$ is U.M.WC.
Proposition 2.8: If $F:(X,\tau) \twoheadrightarrow (Y, \sigma)$ and $G:(X,\tau)\to (Z, \eta)$ are U.M. αC then $F\times G:(X,\tau) \twoheadrightarrow (Y\times Z , \sigma\times\eta)$ is U.M.αC where $\sigma\times\eta$ is the product topology on $Y\times Z$.

Proof: Suppose $F:(X,\tau) \twoheadrightarrow (Y, \sigma)$ and $G:(X,\tau) \twoheadrightarrow (Z , \eta)$ are U.M.αC. Let $x\in X$, $V\in\sigma$ and $W\in\eta$ with $F(x)\times G(x)\subseteq V\times W$. Then $F(x)\subseteq V$ and $G(x)\subseteq W$. Since F and G sre U.M. αC at x, there are α-open sets U_1 and U_2 in X containing x such that $F(x')\cap V\neq\emptyset$ for every $x'\in U_1$ and $G(x'')\cap W\neq\emptyset$ for every $x''\in U_2$. Taking $U = U_1\cap U_2$ we see that $(F(u)\times G(u))\cap (V\times W) \neq\emptyset$ for every $u\in U$. This shows that $F\times G$ is U.M.αC.

Proposition 2.9: If $F: (X,\tau)\to(Y, \sigma)$ and $G: (X,\tau)\to (Z , \eta)$ are U.M. WC then $F\times G: (X,\tau) \twoheadrightarrow (Y\times Z , \sigma\times\eta)$ is U.M. WC.

ISSN: 2350-0328

International Journal of AdvancedResearch in Science, Engineering and Technology

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Proof: Suppose F: $(X,\tau) \twoheadrightarrow (Y, \sigma)$ and G: $(X,\tau) \twoheadrightarrow (Z , \eta)$ are U.M. WC. Let $x\in X$, $V\in\sigma$ and $W\in\eta$ with $F(x)\times G(x)\subseteq V\times W$. Then $F(x)\subseteq V$ and $G(x)\subseteq W$. Since F and G sre U.M.WC at x, there are open sets U_1 and U_2 in X containing x such that $F(x')\cap ClV\neq\emptyset$ for every $x'\in U_1$ and $G(x'')\cap ClW\neq\emptyset$ for every $x''\in U_2$. Taking $U = U_1\cap U_2$ we see that $(F(u)\times G(u))\cap (V\times W) \neq\emptyset$ for every $u\in U$. This shows that $F\times G$ is U.M. WC.

III. Weak forms of lower mixed continuous multifunctions

Definition 3.1: $F:(X, \tau)\twoheadrightarrow (Y, \sigma)$ is

(i)lower mixed weakly continuous (L.M.WC) if for all $x\in X$ and for every $V\in\sigma$ with $F(x) \cap V \neq\emptyset$ there exists $U\in\tau$ containing x such that $F(u) \subseteq ClV$ for every $u\in U$,

(ii) lower mixed b-continuous (L.M.bC) if for all $x\in X$ and for every $V\in\sigma$ with $F(x)\cap V \neq\emptyset$ there exists a b-open set in (X,τ) containing x such that $F(u) \subseteq V$ for every $u\in U$,

(iii) lower mixed α-continuous (L.M.αC) if for all $x\in X$ and for every $V\in\sigma$ with $F(x) \cap V \neq\emptyset$ there exists an α-open set U in (X,τ) containing x such that $F(u) \subseteq V$ for every $u\in U$,

(iv) lower mixed β-continuous (L.M. βC) if for all $x\in X$ and for every $V\in\sigma$ with $F(x)\subseteq V$ there exists a β-open set U in (X,τ) containing x such that $F(u) \subseteq V$ for every $u\in U$,

(v) lower mixed precontinuous (U.M. preC) if for all $x\in X$ and for every $V\in\sigma$ with $F(x) \cap V \neq\emptyset$ there exists a preopen set U in (X,τ) containing x such that $F(u)\subseteq V$ for every $u\in U$.

Since every open set is α-open, β-open, b-open and preopen, it follows from the above definition that if $F:(X, \tau)\twoheadrightarrow (Y, \sigma)$ is lower mixed continuous then it is lower mixed b-continuous, lower mixed α-continuous, lower mixed β-continuous and lower mixed precontinuous functions. Also every lower mixed continuous function is lower mixed weakly continuous. Hence the Definition 3.1 is meaningful as the functions in the definition are the weaker forms of lower mixed continuous functions.

Proposition 3.2 : F: $(X, \tau)\twoheadrightarrow (Y, \sigma)$ is

(i) L.M.αC iff $F^+(V) = F^-(V)$ is α-open in (X, τ) for every open set V in (Y, σ).

(ii) L.M.preC iff $F^+(V)=F^-(V)$ is preopen in (X, τ) for every open set V in (Y, σ).

(iii) L.M.βC iff $F^+(V) = F^-(V)$ is β-closed in (X, τ) for every opent V in (Y, σ).

(iv) L.M.bC iff $F^+(V) = F^-(V)$ is b-open in (X, τ) for every open set V in (Y, σ).

(v) L.M.WC iff $F^-(V)\subseteq F^+(ClV)\subseteq F^-(ClV)$ and $F^+(ClV)$ is open in (X, τ) for $V\in\sigma$.

Proof: Suppose F: $(X, \tau)\twoheadrightarrow (Y, \sigma)$ is L.M.αC. Let $x\in F^-(V)$ and $V\in\sigma$. Then $F(x)\cap V \neq\emptyset$. Since F is L.M.αC at x, there exists $U\in\alpha O(X,\tau)$ containing x such that $F(u) \subseteq V$ for every $u\in U$. This shows that $u\in F^+(V)$ for every $u\in U$ that implies $x\in U\subseteq F^+(V)$ that implies $F^-(V) \subseteq F^+(V)$. Since $F^+(V) \subseteq F^-(V)$, it follows that $F^-(V)= F^+(V)$ is α-open in (X, τ).

Conversely let $F^+(V)= F^-(V)$ be α-open in (X, τ) for every $V\in\sigma$. Suppose $F(x)\cap V \neq\emptyset$. and $V\in\sigma$. Then $x\in F^+(V)= F^-(V)$. This implies that there exists $U\in\alpha O(X,\tau)$ containing x such that $x\in U\subseteq F^-(V)= F^+(V)$. This shows that $u\in F^+(V)$ for every $u\in U$ that implies $F(u) \subseteq V$ for every $u\in U$. This proves that F is L.M.αC. The proof for (ii) ,(iii) and (iv) is similar.

Suppose F: $(X, \tau)\twoheadrightarrow (Y, \sigma)$ is L.M.WC. Let $x\in F^-(V)$ and $V\in\sigma$. Then $F(x)\cap V \neq\emptyset$. Since F is L.M.WC at x, there exists $U\in\tau$ containing x such that $F(u) \subseteq ClV$ for every $u\in U$. This shows that $u\in F^+(ClV)$ for every $u\in U$ that implies $x\in U\subseteq F^+(ClV)$ that implies $F^-(V) \subseteq F^+(ClV) \subseteq F^-(ClV)$ and $F^+(ClV)$ is open in (X, τ).

Conversely let $F^-(V) \subseteq F^+(ClV) \subseteq F^-(ClV)$ and $F^+(ClV)$ is open in (X, τ) for every $V\in\sigma$. Suppose $F(x)\cap V \neq\emptyset$ and $V\in\sigma$. Then $x\in F^-(V)\subseteq F^+(ClV)$. Since $F^+(ClV)$ is open in (X, τ) there exists $U\in\tau$ containing x such that $x\in U\subseteq F^+(ClV)$. This shows that $u\in F^+(ClV)$ for every $u\in U$ that implies $F(u) \subseteq ClV$ for every $u\in U$. This proves that F is L.M.WC. This proves (v).

Proposition 3.3 : F: $(X, \tau)\twoheadrightarrow (Y, \sigma)$ is

(i) L.M.αC iff $F^+(B) = F^-(B)$ is α-closed in (X, τ) for every closed set B in (Y, σ).

(ii) L.M.preC iff $F^+(B)=F^-(B)$ is preclosed in (X,τ) for every closed set B in (Y,σ).

(iii)F is L.M.βC iff $F^+(B) = F^-(B)$ is β-closed in (X, τ) for every closed set B in (Y, σ).

(iv)F is L.M.bC iff $F^+(B) = F^-(B)$ is b-closed in (X, τ) for every closed set B in (Y, σ).

(v)F is L.M.WC iff $F^+(B)\supseteq F^-(IntB) \supseteq F^+(IntB)$ and $F^-(IntB)$ is closed in (X, τ) for every closed set B in (Y, σ).

ISSN: 2350-0328

International Journal of AdvancedResearch in Science, Engineering and Technology

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Proof: F is L.M.αC iff $F^+(Y\setminus B)=F^-(Y\setminus B)$ is α-open in (X, τ) for every closed set B in (Y, σ) iff $X\setminus F^-(B) = X\setminus F^+(B)$ is α-open in (X, τ) for every closed set B in (Y, σ)
iff $F^-(B)=F^+(B)$ is α-closed in (X, τ) for every closed set B in (Y, σ)
This proves (i) and the proof for (ii) ,(iii) and (iv) is analog.
F is L.M.WC iff $F^-(Y\setminus B)\subseteq F^+(Cl(Y\setminus B))\subseteq F^-(Cl(Y\setminus B))$ and $F^+(Cl(Y\setminus B))$ is open in (X, τ) for every closed set B in (Y, σ) iff $X\setminus F^+(B)\subseteq F^+((Y\setminus IntB))\subseteq F^-((Y\setminus IntB))$ and $F^+((Y\setminus IntB))$ is open in (X, τ) for every closed set B in (Y, σ) iff $X\setminus F^+(B)\subseteq X\setminus F^-(IntB)\subseteq X\setminus F^+(IntB)$ and $X\setminus F^-(IntB)$ is open in (X, τ) for every closed set B in (Y, σ) iff $F^+(B)\supseteq F^-(IntB) \supseteq F^+(IntB)$ and $F^-(IntB)$ is closed in (X, τ) for every closed set B in (Y, σ). This proves (v).

Proposition 3.4 : If F:(X, τ)↠ (Y, σ) and G: (X, τ)↠ (Y, σ) are L.M.αC then F∪ G is L.M.αC.
Proof: Suppose F: (X, τ)↠ (Y , σ) and G: (X, τ)↠ (Y , σ) are L.M.αC. Let $x\in X$ and $V\in\sigma$ with $(F\cup G)(x)\cap V\neq\varnothing$. Then $F(x)\cap V\neq\varnothing$ or $G(x)\cap V\neq\varnothing$. If $F(x)\cap V\neq\varnothing$ then there is an α-open set U_1 in X such that $F(x')\subseteq V$ for every $x'\in U_1$. If $G(x)\cap V\neq\varnothing$ then there is an α-open set U_2 in X such that $G(x'')\subseteq V$ for every $x''\in U_2$. Then $U = U_1\cap U_2$ is the required α-open set satisfying $(F\cup G)(u)\subseteq V$ for $u\in U$. This shows the proposition .

Proposition 3.5 : If F:(X, τ)↠ (Y , σ) and G: (X, τ)↠ (Y , σ) are L.M.WC. then F∪ G is L.M.WC.
Proof: Suppose F: (X, τ)↠ (Y , σ) and G: (X, τ)↠ (Y , σ) are L.M.WC. Let $x\in X$ and $V\in\sigma$ with $(F\cup G)(x)\cap V\neq\varnothing$. Then $F(x)\cap V\neq\varnothing$ or $G(x)\cap V\neq\varnothing$. If $F(x)\cap V\neq\varnothing$ then there is an open set U_1 in X such that $F(x')\subseteq V$ for every $x'\in U_1$. If $G(x)\cap V\neq\varnothing$ then there is an open set U_2 in X such that $G(x'')\subseteq V$ for every $x''\in U_2$. Then $U = U_1\cap U_2$ is the required open set satisfying $(F\cup G)(u)\subseteq V$ for $u\in U$. This shows the proposition .

Proposition 3.6: Let G:(Y,σ) ↠ (Z , η) be L.C such that $G(G^-(V)) = V$. Then
(i)If F: (X, τ)↠ (Y,σ) is L.M.αC and then G°F: (X, τ)↠ (Z , η) is L.M.αC .
(ii)If F: (X, τ)↠ (Y,σ) is L.M.preC and then G°F: (X, τ)↠ (Z , η) is L.M.preC .
(iii)If F: (X, τ)↠ (Y,σ) is L.M.βC and then G°F: (X, τ)↠ (Z , η) is L.M.βC .
(iv)If F: (X, τ)↠ (Y,σ) is L.M.βC and then G°F: (X, τ)↠ (Z , η) is L.M.βC .

Proof: Suppose F: (X, τ)↠ (Y,σ) is L.M.αC and G:(Y,σ) ↠ (Z , η) is L.C. Let $x\in X$ and $V\in\eta$ with $G(F(x))\cap V\neq\varnothing$ Then $G(y)\cap V\neq\varnothing$ for every some $y\in F(x)$. Fix this y. Then $y\in G^-(V)$ for every $y\in F(x)$. Since G is L.C, $G^-(V)$ is open in Y. Clearly $F(x)\cap G^-(V)\neq\varnothing$. Since F is L.M.αC, there is an α-open set U in X containing x such that $F(u)\cap G^-(V)\neq\varnothing$ for every $u\in U$. That is $G(F(u)\cap G^-(V)\neq\varnothing$ for every $u\in U$ that implies $G(F(u))\cap G(G^-(V)))\neq\varnothing$ for every $u\in U$. Since $G(G^-(V)))=V$, it follows that $G(F(u))\cap V\neq\varnothing$ for every $u\in U$ that implies G°F is L.M.αC. The proof for (ii) ,(iii) and (iv) is similar.

Proposition 3.7: Let G:(Y,σ)↠ (Z , η) be L.C such that $G(ClV)\subseteq ClG(V)$ and $G(G^-(V)))=V$ for every open set V in (Y,σ). Then if F: (X, τ)↠ (Y,σ) is L.M.WC and then G°F: (X, τ)↠ (Z , η) is L.M.WC
Proof: Suppose F: (X, τ)↠ (Y,σ) is L.M.WC and G:(Y,σ) ↠ (Z , η) is L.C such that $G(ClV)\subseteq ClG(V)$ and $G(G^-(V)))=V$ for every open set V in (Y,σ). Let $x\in X$ and $V\in\eta$ with $G(F(x))\cap V\neq\varnothing$ Then $G(y)\cap V\neq\varnothing$ for every some $y\in F(x)$. Fix this y. Then $y\in G^-(V)$. Since G is L.C, $G^-(V)$ is open in Y. Clearly $F(x)\cap G^-(V)\neq\varnothing$. Since F is L.M.WC, there is an open set U in X containing x such that $F(u)\cap Cl\, G^-(V)\neq\varnothing$ for every $u\in U$. That is $G(F(u)\cap ClG^-(V))\neq\varnothing$ for every $u\in U$ that implies $G(F(u))\cap G(ClG^-(V)))\neq\varnothing$ for every $u\in U$. Since $G(ClV)\subseteq ClG(V)$, $G(F(u))\cap Cl\,(G(G^-(V)))\neq\varnothing$ for every $u\in U$. Again since $G(G^-(V)))=V$, it follows that $G(F(u))\cap Cl\,V\neq\varnothing$ for every $u\in U$ that implies G°F is L.M.WC.

Proposition 3.8: If F: (X,τ) ↠ (Y, σ) and G: (X,τ) ↠ (Z , η) are L.M. αC then F×G: (X,τ) ↠ (Y×Z , σ×η) is L.M. αC where σ×η is the product topology on Y×Z.

Proof: Suppose F: (X,τ) ↠ (Y, σ) and G: (X,τ) ↠ (Z , η) are L.M. αC. Let $x\in X$, $V\in\sigma$ and $W\in\eta$ with $(F(x)\times G(x))\cap(V\times W)\neq\varnothing$. Then $F(x)\cap V\neq\varnothing$ and $G(x)\cap W\neq\varnothing$. Since F and G are L.M. αC at x, there are α-open sets U_1 and U_2 in X containing x such that $F(x')\subseteq V$ for every $x'\in U_1$ and $G(x'')\subseteq W$ for every $x''\in U_2$. Taking $U = U_1\cap U_2$ we see that $(F(u)\times G(u))\subseteq(V\times W)$ for every $u\in U$. This shows that F×G is L.M. αC.

Proposition 3.9: If F: $(X,\tau) \twoheadrightarrow (Y, \sigma)$ and G: $(X,\tau) \twoheadrightarrow (Z , \eta)$ are L.M. WC then $F\times G$: $(X,\tau) \twoheadrightarrow (Y\times Z , \sigma\times\eta)$ is L.M. WC.

Proof: Suppose F: $(X,\tau) \twoheadrightarrow (Y, \sigma)$ and G: $(X,\tau) \twoheadrightarrow (Z , \eta)$ are L.M. WC. Let $x\in X$, $V\in\sigma$ and $W\in\eta$ with $(F(x)\times G(x)) \cap(V\times W) \neq\varnothing$. Then $F(x) \cap V \neq\varnothing$ and $G(x) \cap W \neq\varnothing$. Since F and G are L.M. WC at x, there are pen sets U_1 and U_2 in X containing x such that $F(x')\subseteq ClV$ for every $x'\in U_1$ and $G(x'')\subseteq ClW$ for every $x''\in U_2$. Taking $U = U_1\cap U_2$ we see that $(F(u)\times G(u))\subseteq Cl(V\times W)$ for every $u\in U$. This shows that $F\times G$ is L.M. WC.

Proposition 3.10:

$$\begin{array}{ccccccccc}
U.C & \Rightarrow & U.\alpha C & \Rightarrow & U.preC & \Rightarrow & U.bC. & \Rightarrow & U..\beta C \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
L.M.C & \Rightarrow & L.M.\alpha C & \Rightarrow & L.M.preC & \Rightarrow & L.M.bC. & \Rightarrow & L.M.\beta C \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
U.M.C & \Rightarrow & U.M.\alpha C & \Rightarrow & U.M.preC & \Rightarrow & U.M.bC. & \Rightarrow & U.M.\beta C. \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
L.C & \Rightarrow & L.\alpha C & \Rightarrow & L.preC & \Rightarrow & L.bC. & \Rightarrow & U..\beta C
\end{array}$$

Proof:
The implications L.M.C $\Rightarrow$ L.M.αC $\Rightarrow$ L.M.preC$\Rightarrow$ L.M.bC. $\Rightarrow$ L.M.βC. follow from Definition 3.1. and from the chain: open set $\Rightarrow\alpha$-open set$\Rightarrow$ preopen set $\Rightarrow$b-open set$\Rightarrow\beta$-open set. Also the implications U.M.C $\Rightarrow$ U.M.αC $\Rightarrow$ U.M.preC$\Rightarrow$ U.M.bC. $\Rightarrow$U.M.βC follow from Definition 2.1 and from the above chain.
Let F: $(X, \tau)\twoheadrightarrow (Y, \sigma)$ be lower mixed α-continuous. Suppose $F(x)\subseteq V$ and $V\in\sigma$. Since $F(x) \neq\varnothing$, $F(x)\cap V \neq\varnothing$. Since F is L.M.$\alpha$C, there exists $U\in\alpha O(X,\tau)$ containing x such that $F(U) \subseteq V$. This implies $F(u)\subseteq V$ for every $u\in U$ that further implies that $F(u)\cap V \neq\varnothing$ for every $u\in U$. This shows that F is U.M.αC. This proves that L.M.αC$\Rightarrow$ U.M.αC. Similarly we can establish that L.M.preC$\Rightarrow$ U.M.preC , L.M. bC$\Rightarrow$ U.M. bC and L.M. βC$\Rightarrow$ U.M. βC.

Suppose F is L.M.αC. Let $x\in X$ and $V\in\sigma$ with $F(x)\subseteq V$. Then $F(x)\cap V \neq\varnothing$. Since F is L.M.$\alpha$C at x, there is an $U\in\alpha O(X,\tau)$ containing x such that $F(U)\subseteq V$. This shows that F is U.αC. This proves L.M.αC $\Rightarrow$ U.αC . Now Suppose F is L.αC. Let $x\in X$ and $V\in\sigma$ with $F(x)\subseteq V$ that implies $F(x)\cap V \neq\varnothing$. Since F is L.$\alpha$C, there is an $U\in\alpha O(X,\tau)$ containing x such that $F(u) \cap V \neq\varnothing$ for every $u\in U$. This proves that F is U.M.αC. This proves L.αC $\Rightarrow$ U.M.αC. The other implications can be similarly established.

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