Weak forms of Mixed Continuity

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ABSTRACT: Multivalued functions between topological spaces have applications to the fixed point theory which in turn has applications to social science, science and engineering. The authors have recently studied upper mixed continuous and lower mixed continuous multifunctions between topological spaces. The purpose of this paper is to introduce and characterize some weak forms of upper and lower mixed continuous multifunctions.

KEYWORDS: Multifunctions, upper mixed continuity; lower mixed continuity.

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I. INTRODUCTION AND PRELIMINARIES

Functions between two sets play a significant role in mathematics. In particular multivalued functions have applications in applied mathematics such as optimal control, calculus of variations, probability, statistics, differential inclusions, fixed point theory and so on.

Let X and Y be any two non empty sets. A multivalued function from X to Y, is a function \( F:X \rightarrow \wp(Y) \) such that \( F(x) \neq \emptyset \) for every \( x \in X \) where \( \wp(Y) \) denotes the power set of Y. A multifunction \( F \) from X to Y is denoted by \( F:X \rightrightarrows Y \).

If \( F:X \rightrightarrows Y \) then for every subset A of Y, \( F^+(A) = \{ x \in X : F(x) \subseteq A \} \) is called the upper inverse of A and \( F^-(A) = \{ x \in X : F(x) \cap A \neq \emptyset \} \) is called the lower inverse of A. For every subset A of X, \( F(A) = \bigcup_{x \in A} F(x) \).

For the basic results in multivalued analysis one may consult [2, 7, 9, 12]. The applications of multivalued functions are found in [6, 8, 10, 11, 16, 17, 18, 22, 23, 24, 29]. The properties of multifunctions are discussed in [19, 22, 25]. The continuity of multifunctions is studied in [5, 10, 13, 14, 20, 21, 23, 26, 28, 31-38]. It is easy to see that \( F^+(A) \subseteq F^-(A) \) for every subset A of X. The reverse inclusion is not true. Throughout this paper \( F:X \rightrightarrows Y \) is a multi valued function, A, B are the subsets of X and ‘iff’ denotes ‘if and only if’. The following lemmas will be useful to study the continuity of multifunctions.

Lemma 1.1:

(i) \( F(A \cup B) = F(A) \cup F(B) \) but \( F(A \cap B) \subseteq F(A) \cap F(B) \)

(ii) \( F(X \setminus A) \supseteq F(X) \setminus F(A) \)

(iii) \( A \subseteq B \Rightarrow F(A) \subseteq F(B) \)

Lemma 1.2:

(i) \( F^+(Y \setminus A) = X \setminus F^-(A) \) and \( F^-(Y \setminus A) = X \setminus F^+(A) \)

(ii) \( A \subseteq B \Rightarrow F^+(A) \subseteq F^+(B) \) and \( F^-(A) \subseteq F^-(B) \)

(iii) \( F^+(A \cup B) = F^+(A) \cup F^+(B) \) but \( F^-(A \cap B) \supseteq F^-(A) \cap F^-(B) \)

(iv) \( F^+(A \cup B) \subseteq F^+(A) \cup F^+(B) \) but \( F^-(A \cap B) = F^-(A) \cap F^-(B) \)

(v) If \( u \in F^+(F(x)) \) then \( F(u) \subseteq F(x) \)

(vi) \( F^+(F(A)) \subseteq F(F^-(A)) \)

Definition 1.3: Let \( F:X \rightarrow Y \) and \( G:X \rightarrow Z \). The multifunctions \( F \circ G \) and \( F \cap G \) are defined as \( (F \circ G)(x) = F(x) \cup G(x) \) and \( (F \cap G)(x) = F(x) \cap G(x) \) for every \( x \in X \).

Definition 1.4: Let \( F:X \rightarrow Y \) and \( G:X \rightarrow Z \). The multifunction \( F \times G:X \times Y \rightarrow Z \) is defined as \( (F \times G)(x) = F(x) \times G(x) \) for every \( x \in X \).
Definition 1.5: Let \( F:X \to Y \) and \( G:Y \to Z \). The multifunction \( G^*F:X \to Z \) is defined as \( (G^*F)(x) = \bigcup_{y \in F(x)} G(y) \) for every \( x \in X \).

Lemma 1.6: Let \( F:X \to Y \) and \( G:X \to Y \). Let \( V \subseteq Y \). Then

(i) \( (F \cup G)^*(V) = F^*(V) \cup G^*(V) \) but \( (F \cap G)^*(V) \subseteq F^*(V) \cap G^*(V) \)

(ii) \( (F \cup G)^*(V) \subseteq F^*(V) \cup G^*(V) \) and \( (F \cap G)^*(V) \subseteq F^*(V) \cap G^*(V) \)

The concepts of regular open [27], \( \alpha \)-open [18], semiopen [19], preopen [15], b-open [4] and \( \beta \)-open [1] sets were introduced and studied by Stone, Njastad, Levine, Mashhour et al., Andrijevic and Abdel Monsef et al. respectively. The \( \beta \)-open sets are also called semi-preopen sets in the sense of Andrijevic [3]. The notations \( RO(X, \tau), \alpha O(X, \tau), SO(X, \tau), PO(X, \tau), bO(X, \tau) \) and \( \beta O(X, \tau) \) denote the collection of regular open sets, \( \alpha \)-open sets, semiopen sets, preopen sets, b-open sets and \( \beta \)-open sets in a topological space \( (X, \tau) \) respectively. Similarly the notations \( RC(X, \tau), \alpha C(X, \tau), SC(X, \tau), PC(X, \tau), bC(X, \tau) \) and \( \beta C(X, \tau) \) denote the collection of regular closed sets, \( \alpha \)-closed sets, semiclosed sets, preclosed sets, b-closed sets and \( \beta \)-closed sets in a topological space \( (X, \tau) \) respectively.

Definition 1.7: \( F: (X, \tau) \to (Y, \sigma) \) is

(i) upper continuous (briefly \( U.C \)) if for all \( x \in X \) and for every \( V \subseteq Y \) with \( F(x) \subseteq V \) there exists \( U \subseteq \tau \) containing \( x \) such that \( F(U) \subseteq V \).

(ii) upper weakly continuous (briefly \( U.WC \)) if for all \( x \in X \) and for every \( V \subseteq Y \) with \( F(x) \subseteq V \) there exists an open set \( U \) in \( (X, \tau) \) containing \( x \) such that \( F(U) \subseteq CI V \).

(iii) upper \( \alpha \)-continuous (briefly \( U.\alpha C \)) if for all \( x \in X \) and for every \( V \subseteq Y \) with \( F(x) \subseteq V \) there exists an \( \alpha \)-open set \( U \) in \( (X, \tau) \) containing \( x \) such that \( F(U) \subseteq V \).

(iv) upper precontinuous (briefly \( U.\preC \)) if for all \( x \in X \) and for every \( V \subseteq Y \) with \( F(x) \subseteq V \) there exists a preopen set \( U \) in \( (X, \tau) \) containing \( x \) such that \( F(U) \subseteq V \).

(v) upper \( \beta \)-continuous (briefly \( U.\beta C \)) if for all \( x \in X \) and for every \( V \subseteq Y \) with \( F(x) \subseteq V \) there exists a \( \beta \)-open set \( U \) in \( (X, \tau) \) containing \( x \) such that \( F(U) \subseteq V \).

Definition 1.8: \( F: (X, \tau) \to (Y, \sigma) \) is

(i) lower continuous (briefly \( L.C \)) if for all \( x \in X \) and for every \( V \subseteq Y \) with \( F(x) \cap V \neq \emptyset \) there exists \( U \subseteq \tau \) containing \( x \) such that \( F(u) \cap V \neq \emptyset \) for every \( u \in U \).

(ii) lower weakly continuous (briefly \( L.WC \)) if for all \( x \in X \) and for every \( V \subseteq Y \) with \( F(x) \cap V \neq \emptyset \) there exists an open set \( U \) in \( (X, \tau) \) containing \( x \) such that \( F(u) \cap CV \neq \emptyset \) for every \( u \in U \).

(iii) lower \( \alpha \)-continuous (briefly \( L.\alpha C \)) if for all \( x \in X \) and for every \( V \subseteq Y \) with \( F(x) \cap V \neq \emptyset \) there exists an \( \alpha \)-open set \( U \) in \( (X, \tau) \) containing \( x \) such that \( F(u) \cap V \neq \emptyset \) for every \( u \in U \).

(iv) lower precontinuous (briefly \( L.\preC \)) if for all \( x \in X \) and for every \( V \subseteq Y \) with \( F(x) \cap V \neq \emptyset \) there exists a preopen set \( U \) in \( (X, \tau) \) containing \( x \) such that \( F(u) \cap V \neq \emptyset \) for every \( u \in U \).

(v) lower \( \beta \)-continuous (briefly \( L.\beta C \)) if for all \( x \in X \) and for every \( V \subseteq Y \) with \( F(x) \cap V \neq \emptyset \) there exists a \( \beta \)-open set \( U \) in \( (X, \tau) \) containing \( x \) such that \( F(u) \cap V \neq \emptyset \) for every \( u \in U \).

Definition 1.9: \( F: (X, \tau) \to (Y, \sigma) \) is upper mixed continuous (briefly \( U.M.C \)) [30] if for all \( x \in X \) and for every \( V \subseteq Y \) with \( F(x) \subseteq V \) there exists \( U \subseteq \tau \) containing \( x \) such that \( F(u) \cap V \neq \emptyset \) for every \( u \in U \) and is lower mixed continuous (briefly \( L.M.C \)) [30] if for all \( x \in X \) and for every \( V \subseteq Y \) with \( F(x) \cap V \neq \emptyset \) there exists \( U \subseteq \tau \) containing \( x \) such that \( F(u) \cap V \neq \emptyset \) for every \( u \in U \).
II. Weak forms of upper mixed continuous multifunctions

**Definition 2.1:** \(F:(X, \tau) \rightarrow (Y, \sigma)\) is

(i) upper mixed weakly continuous (briefly U.M.WC) if for all \(x \in X\) and for every \(V \in \sigma\) with \(F(x) \subseteq V\) there exists \(U \in \tau\) containing \(x\) such that \(F(u) \cap CV \neq \emptyset\) for every \(u \in U\),

(ii) upper mixed b-continuous (briefly U.M.bC) if for all \(x \in X\) and for every \(V \in \sigma\) with \(F(x) \subseteq V\) there exists \(U \in \partial O(X, \tau)\) containing \(x\) such that \(F(u) \cap V \neq \emptyset\) for every \(u \in U\),

(iii) upper mixed \(\alpha\)-continuous (briefly U.M.\(\alpha\)C) if for all \(x \in X\) and for every \(V \in \sigma\) with \(F(x) \subseteq V\) there exists \(U \in \partial O(X, \tau)\) containing \(x\) such that \(F(u) \cap V \neq \emptyset\) for every \(u \in U\),

(iv) upper mixed \(\beta\)-continuous (briefly U.M.\(\beta\)C) if for all \(x \in X\) and for every \(V \in \sigma\) with \(F(x) \subseteq V\) there exists \(U \in \partial O(X, \tau)\) containing \(x\) such that \(F(u) \cap V \neq \emptyset\) for every \(u \in U\),

(v) upper mixed precontinuous (briefly U.M. preC) if for all \(x \in X\) and for every \(V \in \sigma\) with \(F(x) \subseteq V\) there exists \(U \in PO(X, \tau)\) containing \(x\) such that \(F(u) \cap V \neq \emptyset\) for every \(u \in U\).

Since every open set is \(\alpha\)-open, \(\beta\)-open, b-open and preopen , it follows from the above definition that if \(F:(X, \tau) \rightarrow (Y, \sigma)\) is upper mixed weakly continuous then it is upper mixed b-continuous, upper mixed \(\alpha\)-continuous. upper mixed \(\beta\)-continuous and upper mixed precontinuous functions. Also every upper mixed continuous function is upper mixed weakly continuous. Hence the Definition 2.1 is meaningful as the functions in the definition are the weaker forms of upper mixed continuous functions.

**Proposition 2.2 :** \(F:(X, \tau) \rightarrow (Y, \sigma)\) is

(i) U.M.\(\alpha\)C iff \(F'(V) \subseteq \alpha Int F(V)\) for every \(V \in \sigma\).

(ii) U.M.preC iff \(F'(V) \subseteq plnt F(V)\) for every \(V \in \sigma\).

(iii) U.M.\(\beta\)C iff \(F'(V) \subseteq \beta Int F(V)\) for every \(V \in \sigma\).

(iv) U.M.\(\beta\)C iff \(F'(V) \subseteq bInt F(V)\) for every \(V \in \sigma\).

(v) U.M.WC iff \(F'(CIV) \subseteq Int F'(CIV)\) for every \(V \in \sigma\).

**Proof:** Suppose \(F:(X, \tau) \rightarrow (Y, \sigma)\) is U.M.\(\alpha\)C. Let \(x \in F'(V)\) and \(V \in \sigma\). Then \(F(x) \subseteq V\). Since \(F\) is U.M.\(\alpha\)C at \(x\), there exists \(U \in \partial O(X, \tau)\) containing \(x\) such that \(F(u) \cap V \neq \emptyset\) for every \(u \in U\). This shows that \(u \in F(V)\) for every \(u \in U\) that implies \(x \in U \subseteq F(V)\). Since \(F'(V) \subseteq F(V)\), the above arguments show that every point of \(F'(V)\) is an \(\alpha\)-interior point of \(F'(V)\) that implies \(F'(V) \subseteq \alpha Int F(V)\).

Conversely let \(F'(V) \subseteq \alpha Int F(V)\) for every \(V \in \sigma\). Suppose \(F(x) \subseteq V\) and \(V \in \sigma\). Then \(x \in F'(V) \subseteq \alpha Int F'(V)\). This implies that there exists \(U \in \partial O(X, \tau)\) containing \(x\) such that \(x \in U \subseteq F'(V)\). This shows that \(u \in F'(V)\) for every \(u \in U\) that implies \(F(u) \cap V \neq \emptyset\) for every \(u \in U\). This proves that \(F \) is U.M.\(\alpha\)C. The proof for (ii) , (iii) and (iv) is similar.

Suppose \(F:(X, \tau) \rightarrow (Y, \sigma)\) is U.M.WC. Let \(x \in F'(V)\) and \(V \in \sigma\). Then \(F(x) \subseteq V\). Since \(F\) is U.M.WC at \(x\), there exists \(U \in \tau\) containing \(x\) such that \(F(u) \cap CIV \neq \emptyset\) for every \(u \in U\). This shows that \(u \in F'(CIV)\) for every \(u \in U\) that implies \(x \in U \subseteq F'(CIV)\). Since \(F'(CIV) \subseteq F'(CIV)\), the above arguments show that every point of \(F'(CIV)\) is an interior point of \(F'(CIV)\) that implies \(F'(CIV) \subseteq Int F'(CIV)\).

Conversely let \(F'(CIV) \subseteq Int F'(CIV)\) for every \(V \in \sigma\). Suppose \(F(x) \subseteq V\) and \(V \in \sigma\). Then \(x \in F'(V) \subseteq F'(CIV) \subseteq Int F'(CIV)\). This implies that there exists \(U \in \tau\) containing \(x\) such that \(x \in U \subseteq F'(CIV)\). This shows that \(u \in F'(CIV)\) for every \(u \in U\) that implies \(F(u) \cap CIV \neq \emptyset\) for every \(u \in U\). This proves that \(F \) is U.M.WC. This proves (v).

**Proposition 2.3 :** \(F:(X, \tau) \rightarrow (Y, \sigma)\) is

(i) U.M.\(\alpha\)C iff \(\alpha CIF'(V) \subseteq F'(V)\) for every closed set \(V\) in \((Y, \sigma)\).

(ii) U.M.preC iff \(p CIF'(V) \subseteq F'(V)\) for every closed set \(V\) in \((Y, \sigma)\).

(iii) U.M.\(\beta\)C iff \(\beta CIF'(V) \subseteq F'(V)\) for every closed set \(V\) in \((Y, \sigma)\).

(iv) U.M.\(\beta\)C iff \(b CIF'(V) \subseteq F'(V)\) for every closed set \(V\) in \((Y, \sigma)\).

(v) U.M.WC iff \(Cl F'(IntB) \subseteq F'(IntB)\) for every closed set \(B\) in \((Y, \sigma)\).

**Proof:** \(F \) is U.M.\(\alpha\)C iff \(F'(Y) \subseteq \alpha Int F'(Y)\) for every closed set \(B\) in \((Y, \sigma)\).

\[i f \ X \setminus F'(B) \subseteq \alpha Int (X \setminus F'(B)) = X \setminus Cl F'(B)\]

\[i f \ \alpha Cl F'(B) \subseteq F'(B)\] for every closed set \(B\) in \((Y, \sigma)\).
This proves (i) and the proof for (ii) , (iii) and (iv) is analog.

F is U.M.WC iff \( F'( (Y \cup B)) \subseteq \text{Int} F' ( (Y \cup B)) \) for every closed set \( B \) in \( (Y, \sigma) \).

iff \( F'( (Y \cup B)) \subseteq \text{Int} F' ( (Y \cup B)) \)

iff \( X \cap F'( (Y \cup B)) \subseteq \text{Int} (X \cap F'( (Y \cup B))) = X \cap F' ( (Y \cup B)) \)

iff \( Cl (F' ( (Y \cup B))) \subseteq F' ( (Y \cup B)) \) for every closed set \( B \) in \( (Y, \sigma) \).

This proves (v).

**Proposition 2.4**: If \( F: (X, \tau) \rightarrow (Y, \sigma) \) and \( G: (X, \tau) \rightarrow (Y, \sigma) \) are U.M. WC then \( F \times G \) is U.M. WC.

**Proof**: Suppose \( F: (X, \tau) \rightarrow (Y, \sigma) \) and \( G: (X, \tau) \rightarrow (Y, \sigma) \) are U.M. WC. Let \( x \in X \) and \( V \in \sigma \) with \( (F \times G)(x) \subseteq V \). Then \( F(x) \subseteq V \) and \( G(x) \subseteq V \). Since \( F: (X, \tau) \rightarrow (Y, \sigma) \) and \( G: (X, \tau) \rightarrow (Y, \sigma) \) are U.M. WC at \( x \), there are \( \alpha \)-open sets \( U_1 \) and \( U_2 \) in \( X \) such that \( F(x) \cap U_1 \neq \emptyset \) for every \( x' \in U_1 \) and \( G(x') \cap V \neq \emptyset \) for every \( x'' \in U_2 \). Then \( U = U_1 \cap U_2 \) is the required \( \alpha \)-open set satisfying \( (F \times G)(u) \cap V \neq \emptyset \) for \( u \in U \).

**Proposition 2.5**: If \( F: (X, \tau) \rightarrow (Y, \sigma) \) and \( G: (X, \tau) \rightarrow (Y, \sigma) \) are U.M.WC then \( F \cup G \) is U.M.WC.

**Proof**: Suppose \( F: (X, \tau) \rightarrow (Y, \sigma) \) and \( G: (X, \tau) \rightarrow (Y, \sigma) \) are U.M.WC. Let \( x \in X \) and \( V \in \sigma \) with \( (F \cup G)(x) \subseteq V \). Then \( F(x) \subseteq V \) and \( G(x) \subseteq V \). Since \( F: (X, \tau) \rightarrow (Y, \sigma) \) and \( G: (X, \tau) \rightarrow (Y, \sigma) \) are U.M.WC at \( x \), there are open sets \( U_1 \) and \( U_2 \) in \( X \) such that \( F(x) \cap U_1 \neq \emptyset \) for every \( x' \in U_1 \) and \( G(x') \cap V \neq \emptyset \) for every \( x'' \in U_2 \). Then \( U = U_1 \cap U_2 \) is the required open set satisfying \( (F \cup G)(u) \cap V \neq \emptyset \) for \( u \in U \).

**Proposition 2.6**: Let \( G: (Y, \sigma) \rightarrow (Z, \eta) \) be U.C.

(i) If \( F: (X, \tau) \rightarrow (Y, \sigma) \) is U.M.C and then \( G \circ F: (X, \tau) \rightarrow (Z, \eta) \) is U.M.C.

(ii) If \( F: (X, \tau) \rightarrow (Y, \sigma) \) is U.M.C and then \( G \circ F: (X, \tau) \rightarrow (Z, \eta) \) is U.M.C.

(iii) If \( F: (X, \tau) \rightarrow (Y, \sigma) \) is U.M.C and then \( G \circ F: (X, \tau) \rightarrow (Z, \eta) \) is U.M.C.

(iv) If \( F: (X, \tau) \rightarrow (Y, \sigma) \) is U.M.C and then \( G \circ F: (X, \tau) \rightarrow (Z, \eta) \) is U.M.C.

**Proof**: Suppose \( F: (X, \tau) \rightarrow (Y, \sigma) \) is U.M.C and \( G: (Y, \sigma) \rightarrow (Z, \eta) \) is U.C. Let \( x \in X \) and \( V \in \eta \) with \( G(F)(x) \subseteq V \). Then \( G(y) \subseteq V \) for every \( y \in F(x) \). That is \( y \in G(F)(x) \) for every \( y \in F(x) \). Since \( G \) is U.C., \( G(F)(x) \) is open in \( Y \). Clearly \( F(x) \subseteq G(F)(x) \). Since \( F \) is U.M.C, there is an \( \alpha \)-open set \( U \) in \( X \) containing \( x \) such that \( F(u) \cap G(F)(x) \neq \emptyset \) for every \( u \in U \). That is \( G(F)(u) \cap G(F)(x) \neq \emptyset \) for every \( u \in U \) that implies \( G(F(u)) \cap G(F(x)) \neq \emptyset \) for every \( u \in U \). This shows that \( G(F)(u) \cap V \neq \emptyset \) for every \( u \in U \) for which \( G(F)(u) \cap G(F(x)) \neq \emptyset \) for every \( u \in U \).

**Proposition 2.7**: Let \( G: (Y, \sigma) \rightarrow (Z, \eta) \) be U.C such that \( G(CV) \subseteq CG \) for every open set \( V \) in \( (Y, \sigma) \). Then if \( F: (X, \tau) \rightarrow (Y, \sigma) \) is U.M.WC and then \( G \circ F: (X, \tau) \rightarrow (Z, \eta) \) is U.M.WC.

**Proof**: Suppose \( F: (X, \tau) \rightarrow (Y, \sigma) \) is U.M.WC and \( G: (Y, \sigma) \rightarrow (Z, \eta) \) is U.C. Let \( x \in X \) and \( V \in \eta \) with \( G(F)(x) \subseteq V \). Then \( G(y) \subseteq V \) for every \( y \in F(x) \). That is \( y \in G(F)(x) \) for every \( y \in F(x) \). Since \( G \) is U.C., \( G(F)(x) \) is open in \( Y \). Clearly \( F(x) \subseteq G(F)(x) \). Since \( F \) is U.M.WC, there is an open set \( U \) in \( X \) containing \( x \) such that \( F(u) \cap G(F)(x) \neq \emptyset \) for every \( u \in U \). That is \( G(F)(u) \cap Cl(G(F)(x)) \neq \emptyset \) for every \( u \in U \) which implies \( G(F(u)) \cap G(CV) \neq \emptyset \) for every \( u \in U \). This implies \( G(F), G(CV) \) is open in \( Z \).

**Proposition 2.8**: If \( F: (X, \tau) \rightarrow (Y, \sigma) \) and \( G: (X, \tau) \rightarrow (Z, \eta) \) are U.M. C then \( F \times G: (X, \tau) \rightarrow (Y \times Z, \sigma \times \eta) \) is U.M. C where \( \sigma \times \eta \) is the product topology on \( Y \times Z \).

**Proof**: Suppose \( F(x, \tau) \rightarrow (Y, \sigma) \) and \( G(x, \tau) \rightarrow (Z, \eta) \) are U.M. C. Let \( x \in X \), \( V \in \sigma \) and \( W \in \eta \) with \( F(x) \subseteq V \times W \). Then \( F(x) \subseteq V \) and \( G(x) \subseteq W \). Since \( F \) and \( G \) are U.M. C at \( x \), there are \( \alpha \)-open sets \( U_1 \) and \( U_2 \) in \( X \) containing \( x \) such that \( F(x) \cap V \neq \emptyset \) for every \( x' \in U_1 \) and \( G(x') \cap W \neq \emptyset \) for every \( x'' \in U_2 \). Taking \( U = U_1 \cap U_2 \) we see that \( (F(x) \times G(x)) \cap (V \times W) \neq \emptyset \) for every \( u \in U \). This shows that \( F \times G \) is U.M. C.

**Proposition 2.9**: If \( F: (X, \tau) \rightarrow (Y, \sigma) \) and \( G: (X, \tau) \rightarrow (Z, \eta) \) are U.M. WC then \( F \times G: (X, \tau) \rightarrow (Y \times Z, \sigma \times \eta) \) is U.M. WC.
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Proposition 3.2 : F: (X, τ)→ (Y, σ) is
(i) L.M.αC iff F'(V) = F(V) is α-open in (X, τ) for every open set V in (Y, σ).
(ii) L.M.preC iff F'(V)=F (V) is preopen in (X, τ) for every open set V in (Y, σ).
(iii) L.M.βC iff F'(V) = F'(V) is β-closed in (X, τ) for every open V in (Y, σ).
(iv) L.M.βC iff F'(V) = F'(V) is b-open in (X, τ) for every open V in (Y, σ).
(v) L.M.WC iff F'(V)=F'(C(V))= F'(C(V)) and F'(C(V)) is open in (X, τ) for V∈σ.

Proof: Suppose F: (X, τ)→ (Y, σ) and G: (X, τ)→ (Z, η) are U.M. WC. Let x ∈ X, V∈σ and W∈η with F(x)×G(x)≤V×W. Then F(x)≤V and G(x)≤W. Since F and G are U.M.WC at x, there are open sets U₁ and U₂ in X containing x such that F(x)∩C(V)≠Ø for every x' ∈ U₁ and G(x')∩W≠Ø for every x' ∈ U₂. Taking U = U₁∩ U₂ we see that (F(u)×G(u))∩ (V×W)≠Ø for every u∈U. This shows that F× G is U.M. WC.

III. Weak forms of lower mixed continuous multifunctions

Definition 3.1: F: (X, τ)→ (Y, σ) is
(i) lower mixed weakly continuous (L.M.WC) if for all x∈X and for every V∈σ with F(x)∩V ≠Ø there exists U∈τ containing x such that F(u)≤C(V) for every u∈U.
(ii) lower mixed b-continuous (L.M.bC) if for all x∈X and for every V∈σ with F(x)∩V ≠Ø there exists a b-open set in (X, τ) containing x such that F(u)≤V for every u∈U.
(iii) lower mixed α-continuous (L.M.αC) if for all x∈X and for every V∈σ with F(x)∩V ≠Ø there exists an α-open set U in (X, τ) containing x such that F(u)≤V for every u∈U.
(iv) lower mixed β-continuous (L.M. βC) if for all x∈X and for every V∈σ with F(x)≤V there exists a β-open set U in (X, τ) containing x such that F(u)≤V for every u∈U.
(v) lower mixed precontinuous (U.M. preC) if for all x∈X and for every V∈σ with F(x)∩V ≠Ø there exists a preopen set U in (X, τ) containing x such that F(u)≤V for every u∈U.

Since every open set is α-open, β-open, b-open and preopen, it follows from the above definition that if F: (X, τ)→ (Y, σ) is lower mixed continuous then it is lower mixed b-continuous, lower mixed α-continuous, lower mixed β-continuous and lower mixed precontinuous functions. Also every lower mixed continuous function is lower mixed weakly continuous. Hence the Definition 3.1 is meaningful as the functions in the definition are the weaker forms of lower mixed continuous functions.

Proposition 3.3 : F: (X, τ)→ (Y, σ) is
(i) L.M.αC iff F'(B) = F'(B) is α-closed in (X, τ) for every closed set B in (Y, σ).
(ii) L.M.preC iff F'(B) = F'(B) is preclosed in (X, τ) for every closed set B in (Y, σ).
(iii) F is L.M.βC iff F'(B) = F'(B) is β-closed in (X, τ) for every closed set B in (Y, σ).
(iv) if F is L.M.bC iff F'(B) = F'(B) is b-closed in (X, τ) for every closed set B in (Y, σ).
(v) F is L.M.WC iff F'(B)=F'(IntB)≥F'(IntB) and F'(IntB) is closed in (X, τ) for every closed set B in (Y, σ).

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Proof: If $F$ is L.M.$\alpha$C iff $F(Y\cap B)=F(Y\setminus B)$ is $\alpha$-open in $(X, \tau)$ for every closed set $B$ in $(Y, \sigma)$ iff $X\setminus F'(B)=X\setminus F'(B)$ is $\alpha$-open in $(X, \tau)$ for every closed set $B$ in $(Y, \sigma)$

This proves (i) and the proof for (ii), (iii) and (iv) is analogous.

Proposition 3.4: If $F(X, \tau)\rightarrow (Y, \sigma)$ and $G:(X, \tau)\rightarrow (Y, \sigma)$ are L.M.$\alpha$C, then $F\cup G$ is L.M.$\alpha$C.

Proof: Suppose $F:(X, \tau)\rightarrow (Y, \sigma)$ and $G:(X, \tau)\rightarrow (Y, \sigma)$ are L.M.$\alpha$C. Let $x\in X$ and $V\in \sigma$ with $(F\cup G)(x)\cap V\neq \emptyset$. Then $(F\cup G)(x)\cap V\neq \emptyset$. If $F(x)\cap V\neq \emptyset$ then there is an open set $U_1$ in $X$ such that $F(x)^\prime \subseteq V$ for every $x^\prime \in U_1$. If $G(x)\cap V\neq \emptyset$ then there is an open set $U_2$ in $X$ such that $G(x)^\prime \subseteq V$ for every $x^\prime \in U_2$. Then $U = U_1 \cap U_2$ is the required open set satisfying $(F\cup G)(u)\subseteq V$ for $u\in U$. This shows the proposition.

Proposition 3.5: If $F:(X, \tau)\rightarrow (Y, \sigma)$ and $G:(X, \tau)\rightarrow (Y, \sigma)$ are L.M.WC, there is an open set $U$ in $X$ containing $x$ such that $F(u)\cap V\neq \emptyset$ for every $u \in U$. This shows the proposition.

Proposition 3.6: Let $G:(Y, \sigma)\rightarrow (Z, \eta)$ be L.C such that $G(G(V))=V$. Then

(i) If $F:(X, \tau)\rightarrow (Y, \sigma)$ is L.M.$\alpha$C and then $G\circ F:(X, \tau)\rightarrow (Z, \eta)$ is L.M.$\alpha$C.

(ii) If $F:(X, \tau)\rightarrow (Y, \sigma)$ is L.M.preC and then $G\circ F:(X, \tau)\rightarrow (Z, \eta)$ is L.M.preC.

(iii) If $F:(X, \tau)\rightarrow (Y, \sigma)$ is L.M.$\beta$C and then $G\circ F:(X, \tau)\rightarrow (Z, \eta)$ is L.M.$\beta$C.

Proof: Suppose $F:(X, \tau)\rightarrow (Y, \sigma)$ is L.M.$\alpha$C and $G:(Y, \sigma)\rightarrow (Z, \eta)$ is L.C. Let $x\in X$ and $V\in \eta$ with $G(F(x))\cap V\neq \emptyset$. Then $G(F(x))\cap V\neq \emptyset$ for every some $y\in F(x)$. Fix this $y$. Then $y\in G(V)$ for every $y\in F(x)$. Since $G$ is L.C, $G(V)$ is open in $Y$. Clearly $F(x)\cap G(V)\neq \emptyset$. Since $F$ is L.M.$\alpha$C, there is an open set $U$ in $X$ containing $x$ such that $F(u)\cap G(V)\neq \emptyset$ for every $u\in U$. That is $G(F(u)\cap G(V))\neq \emptyset$ for every $u\in U$ that implies $G(F(u))\cap G(G(V))\neq \emptyset$ for every $u\in U$. Since $G(G(V))=V$, it follows that $G(F(u))\cap V\neq \emptyset$ for every $u\in U$ that implies $G\circ F$ is L.M.$\alpha$C. The proof for (ii), (iii) and (iv) is similar.

Proposition 3.7: Let $G:(Y, \sigma)\rightarrow (Z, \eta)$ be L.C such that $G(C(V))\subseteq C(G(V))$ and $G(G(V))=V$ for every open set $V$ in $(Y, \sigma)$. Then if $F:(X, \tau)\rightarrow (Y, \sigma)$ is L.M.$\alpha$C and then $G\circ F:(X, \tau)\rightarrow (Z, \eta)$ is L.M.$\beta$C.

Proof: Suppose $F:(X, \tau)\rightarrow (Y, \sigma)$ is L.M.$\alpha$C and then $G\circ F:(X, \tau)\rightarrow (Y, \sigma)$ is L.M.$\beta$C. Let $x\in X$ and $V\in \eta$ with $G(F(x))\cap V\neq \emptyset$. Then $G(F(x))\cap V\neq \emptyset$ for every some $y\in F(x)$. Fix this $y$. Then $y\in G(V)$. Since $G$ is L.C, $G(V)$ is open in $Y$. Clearly $F(x)\subseteq G(V)$ is open in $Y$. Since $F$ is L.M.$\alpha$C, there is an open set $U$ in $X$ containing $x$ such that $F(u)\subseteq G(V)$ is open for every $u\in U$. That is $G(F(u)\subseteq G(G(V)))\subseteq V$ for every $u\in U$. Again since $G(G(V))=V$, it follows that $G(F(u))\subseteq G(V)$ is open for every $u\in U$ that implies $G\circ F$ is L.M.$\beta$C.

Proposition 3.8: If $F:(X, \tau)\rightarrow (Y, \sigma)$ and $G:(X, \tau)\rightarrow (Z, \eta)$ are L.M.$\alpha$C then $F\times G:(X, \tau)\rightarrow (Y\times Z, \sigma\times \eta)$ is L.M.$\alpha$C where $\sigma\times \eta$ is the product topology on $Y\times Z$.

Proof: Suppose $F:(X, \tau)\rightarrow (Y, \sigma)$ and $G:(X, \tau)\rightarrow (Z, \eta)$ are L.M.$\alpha$C. Let $x\in X$, $V\in \sigma$ and $W\in \eta$ with $(F(x)\times G(x))\cap (V\times W)\neq \emptyset$. Then $F(x)\subseteq V$ and $G(x)\subseteq W$. Since $F$ and $G$ are L.M.$\alpha$C at $x$, there are $\alpha$-open sets $U_1$ and $U_2$ in $X$ containing $x$ such that $F(x)^\prime \subseteq V$ for every $x^\prime \in U_1$ and $G(x)^\prime \subseteq W$ for every $x^\prime \in U_2$. Taking $U = U_1 \cap U_2$ we see that $(F(x)\times G(x))\subseteq (V\times W)$ for every $u\in U$. This shows that $F\times G$ is L.M.$\alpha$C.
Proposition 3.9: If F: (X,τ) → (Y, σ) and G: (X,τ) → (Z , η) are L.M. WC then F×G: (X,τ) → (Y×Z , σ×η) is L.M. WC.

Proof: Suppose F: (X,τ) → (Y, σ) and G: (X,τ) → (Z , η) are L.M. WC. Let x ∈ X , V ∈ σ and W ∈ η with (F(x))×G(x) ∩ (V×W) ≠ ∅. Then F(x) ∩ V ≠ ∅ and G(x) ∩ W ≠ ∅. Since F and G are L.M. WC at x, there are pen sets U₁ and U₂ in X containing x such that F(x) ⊆ Cl(V×W) for every x₁ ∈ U₁ and G(x₂) ⊆ Cl(W×V) for every x₂ ∈ U₂. Taking U = U₁ ∩ U₂ we see that (F(x)×G(x)) ⊆ Cl(V×W) for every x ∈ U. This shows that F×G is L.M. WC.

Proposition 3.10:

∧ ∧ ∧ ∧
∧ ∧ ∧ ∧
∧ ∧ ∧ ∧
L.C ⇒ L.α.C ⇒ L.bC. ⇒ U.β.C
∧ ∧ ∧ ∧

Proof:
The implications L.M.C ⇒ L.M.α.C ⇒ L.M.preC ⇒ L.M.bC. ⇒ L.M.β.C. follow from Definition 3.1. and from the chain: open set ⇒ α-open set⇒ preopen set⇒ b-open set⇒ β-open set. Also the implications U.M.C ⇒ U.M.α.C ⇒ U.M.preC ⇒ U.M.bC. ⇒ U.M.β.C follow from Definition 2.1 and from the above chain. Let F: (X,τ) → (Y, σ) be lower mixed α-continuous. Suppose F(x) ⊆ V and V ∈ σ. Since F(x) ≠ ∅, F(x) ∩ V ≠ ∅. Since F is L.M.α.C, there exists U ∈ O(X,τ) containing x such that F(U) ⊆ V. This implies F(u) ⊆ V for every u ∈ U that further implies that F(u)∩V ≠ ∅ for every u ∈ U. This shows that F is U.M.α.C. This proves that L.M.α.C ⇒ U.M.α.C. Similarly we can establish that L.M.preC ⇒ U.M.preC, L.M. bC ⇒ U.M. bC and L.M. βC ⇒ U.M. βC.

Suppose F is L.M.α.C. Let x ∈ X and V ∈ σ with F(x) ⊆ V. Then F(x)∩V ≠ ∅. Since F is L.M.α.C at x, there is an U ∈ O(X,τ) containing x such that F(U) ⊆ V. This shows that F is U.α.C. This proves L.M.α.C ⇒ U.α.C. Now suppose F is L.α.C. Let x ∈ X and V ∈ σ with F(x) ⊆ V that implies F(x)∩V ≠ ∅. Since F is L.α.C, there is an U ∈ O(X,τ) containing x such that F(U)∩V ≠ ∅ for every u ∈ U. This proves that F is U.M.α.C. This proves L.α.C ⇒ U.α.C.
The other implications can be similarly established.

REFERENCES


