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Approximate Solution for SIR Epidemic Model by Generalized Mittag-Leffler Function Method

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ABSTRACT: Through this article, we applied the Generalized Mittag-Leffler Function Method (GMLFM) to find the solutions of epidemic model of a vector-born disease that is supposed to have a constant size over the period of the epidemic. The Generalized Mittag-Leffler function method have already proved their efficiency as solutions in the fractional context, by means of a technique developed for strongly fractional non-linear problems. Fractional derivatives are depict by Caputo sense. In this paper, we comparing the results that we obtained by using (GMLFM) with the results obtained by Runge-Kutta (RK4) method in case of integer order derivative.

KEYWORDS: Fractional calculus; Fractional non-linear system; Mittag-Leffler function; Epidemic model; SIR model.

I.INTRODUCTION

The study of a fractional order epidemic model of a vector-born disease is considered as the main challenge in this work. A vector-born disease has been discussed in many previous studies [1, 9, 16] and all of them defined it as a type of infectious contagious disease that can be produced by bacteria, viruses, protozoa or rickets which are predominately transferred by diseases transferring biological agents "anthropoids", which carry the disease without getting it themselves and dengue are considered the most important models that related to a specific vector-born disease in the recent years.

The models of Mathematical have been used to explain the prevalence of vector-borne disease in the host people. Many models had been applied in this field started with D.Bernoulli (1766) that generate to modern epidemiology then the beginning of 20th century another contagious disease model offered by Ross and Hamer who used mass action law to explain the conduct of epidemic models.

A deterministic model formulated by A.G.Mckendrick and W.O.Kermack in 1927 [6] who described by "Susceptible-Infected-Recovered" that called "SIR model" or "XYZ model". The following non-linear system of ordinary differential equations are described the spread of diseases gives by

$$\begin{aligned} \frac{du(t)}{dt} &= -\beta u(t)v(t), \\ \frac{dv(t)}{dt} &= \beta u(t)v(t) - \gamma v(t), \\ \frac{dw(t)}{dt} &= \gamma v(t), \end{aligned} \tag{1}$$

where

$u(t)$: The number of susceptible individuals,
$v(t)$: The number of infected individuals at time t,
$w(t)$: The number of isolated individuals,
β : The rate of infection,

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γ : The rate at which current infective population become isolated.

Many researchers working on this system, but in formula of fractional differential equations such as [4, 5, 10, 11, 13]. The dynamic of disease in the host population can be described by the following ordinary differential equation "ODE" model which can be interplead the spread of a vector-transmitted in a host population [16]

$$\begin{aligned}\frac{ds(t)}{dt} &= b - \lambda s(t)i(t) - \lambda_1 s(t)v(t) - \mu s(t),\\ \frac{di(t)}{dt} &= \lambda s(t)i(t) + \lambda_1 s(t)v(t) - \gamma i(t) - \mu i(t),\\ \frac{dr(t)}{dt} &= -\gamma i(t) - \mu r(t),\end{aligned} \tag{2}$$

where
$s(t)$: The size of susceptible subclass,
$i(t)$: The size of infectious subclass,
$r(t)$: The size of recovered subclass,
$v(t)$: Number of vectors at time t who carry the pathogen,
b : The host population is recruited a rate,
λ : The rate of direct transmission,
λ_1 : The biting rate that a pathogen-carrier vector has of susceptible hosts,
μ : The host population dies at a Nature death rate,
γ : The per capita recovery rate of the hosts.
"Thieme" [15] is pointed out that the previous dynamic of system (2) is qualitatively equivalent to the dynamics of system given by

$$\begin{aligned}\frac{ds(t)}{dt} &= b - \lambda s(t)i(t) - \lambda_1 s(t)v(t) - \mu s(t),\\ \frac{di(t)}{dt} &= \lambda s(t)i(t) + \lambda_1 s(t)v(t) - \gamma i(t) - \mu i(t),\\ \frac{dv(t)}{dt} &= \lambda_2\left(\frac{b_1}{\mu_1} - v(t)\right) - \mu_1 v(t),\end{aligned} \tag{3}$$

where
λ_2 : Susceptible vectors start carrying the pathogen after getting into contact (biting) an infective host at a rate,
b_1 : The vector population is recruited at a birth rate,
μ_1 : The vector population dies at a natural death rate.

Applications of differential equation, specially fractional differential equations have become widely sold in many life fields such as biology, engineering, fluid mechanics, viscoelastic and physics epidemiology [7,14]. The basic features for the solutions of fractional differential equations reflect the accuracy and reality of different theories which represent them that through they take into account the history of previous time plus the instant time. Recently, most of mathematicians consider that the fracture calculus are one of the most valuable tools to model linear and non-linear problems because they provide prompt and lucid symbolic of analytic solutions, as well as numerical approximate solutions without linearization or non-linearization. Now, in this paper we debate fractional order so as to show the realistic biphasic slope behavior of contagion of disease for a slow rate. Then the formula of fractional model is defined by

$$\begin{aligned}{}^{C}D^{\alpha}x(t) &= b_1 - j_1 x(t)y(t) - j_2 x(t)z(t) - f_1 x(t),\\ {}^{C}D^{\alpha}y(x) &= j_1 x(t)y(t) + j_2 x(t)z(t) - (b_3 + f_1)y(t),\\ {}^{C}D^{\alpha}z(x) &= j_3\left(\frac{b_2}{f_2} - z(t)\right)y(t) - f_2 z(t),\end{aligned} \tag{4}$$

where

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$x = s, y = i, z = v, x(0) = N_1 = 3, y(0) = N_2 = 4, z(0) = N_3 = 5, b_1 = b = 0.1, b_2 = b_1 = 0.5, b_3 = \gamma = 0.3, j_1 = \lambda = 0.2, j_2 = \lambda_1 = 0.4, j_3 = \lambda_2 = 0.6, f_1 = \mu = 0.01, f_2 = \mu_1 = 0.02, 0 < \alpha \leq 1, and\ s + i + v = x + y + z = N,$ where N show the total number of the individuals.

We organized the paper as follows. In Section 2, we show some definitions and notations related to fractional calculus, and then we discussed the using automatic "GMLF" method in Section 3. Last but not least we review the numerical analysis results in Section 4. Finally in Section 5, we discuss the result that we obtained it and comparing with the another method.

II. PRELIMINARY

We present some necessary definitions and notations related to fractional calculus [8] :

Definition 2.1. The fractional integral of order $\alpha > 0$ in the Riemann-Liouville sense is defined as

$${}_aI_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - \xi)^{\alpha-1} f(\xi) d\xi \,, \quad \alpha \geq 0 \,, \quad x > a \,,$$

$${}_aI_x^0 f(x) = f(x),$$

where $\Gamma(\alpha)$ is Eular Gamma function.

Definition 2.2. The Caputo fractional derivative of order α

$${}_a^CD_x^\alpha f(x) = \frac{1}{\Gamma(n - \alpha)} \int_a^x (x - \xi)^{n-\alpha-1} f^{(n)}(\xi) d\xi \,, \quad n - 1 < \alpha < n \,, \quad x > a.$$

At $\alpha = 1$, $D = \frac{d}{dt}$

Now, we review some properties of fractional calculus and for more details see [17, 18]

$${}_a^CD_x^\alpha \; {}_aI_x^\alpha \; f(x) = f(x),$$

$${}_aI_x^\alpha \; {}_a^CD_x^\alpha \; f(x) = f(x) - \sum_{k=0}^{n-1} f^k(a) \frac{(x - a)^k}{\Gamma(k + 1)}.$$

III. ANALYSIS OF METHOD

The Mittag-Leffler (1902-1905) functions E_α and $E_{\alpha,\beta}$ defined by the power series

$$E_\alpha(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n\alpha + 1)} \,, \quad E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n\alpha + \beta)} \,, \quad \alpha, \beta > 0 \,. \qquad (5)$$

have already proved their eligibility as solutions of fractional order differential and integral equations and then have become significant elements of the theory of fractional calculus and its applications [8].

In this paper, we show how to solve system of non-linear fractional differential equations (SIR system) through the imposition of the generalized Mittag-Leffler function. The generalized Mittag-Leffler function method suggests that $y_i(t)$, i = 1,2,3,.... are decomposed by an infinite series of components [2, 3,12]:

$$y_i(t) = E_\alpha(a_i t^\alpha) = \sum_{n=0}^{\infty} a_i^n \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} \,, \quad i = 1,2,3, \dots \,. \qquad (6)$$

We use the following definitions of fractional calculus:

$${}^CD^\alpha y_i(t) = \sum_{n=1}^{\infty} a_i^n \frac{t^{(n-1)\alpha}}{\Gamma((n - 1)\alpha + 1)} \,, \quad i = 1,2,3, \dots \,. \qquad (7)$$

This is established on the Caputo fractional derivatives. The convergence of the Mittag-Leffler function is show in [8].

IV. APPLICATION AND RESULTS

In this section, we will show how to solve system (4) by using "GMLFM" .

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Let,

$$x=\sum_{n=0}^{\infty}a^{n}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)},$$
$$y=\sum_{n=0}^{\infty}d^{n}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)},\qquad(8)$$
$$z=\sum_{n=0}^{\infty}l^{n}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}.$$

By using Caputo fraction derivative we have,

$${}^{C}D^{\alpha}x=\sum_{n=1}^{\infty}a^{n}\frac{t^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)}=\sum_{n=0}^{\infty}a^{n+1}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)},$$
$${}^{C}D^{\alpha}y=\sum_{n=1}^{\infty}d^{n}\frac{t^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)}=\sum_{n=0}^{\infty}d^{n+1}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)},\qquad(9)$$
$${}^{C}D^{\alpha}z=\sum_{n=1}^{\infty}l^{n}\frac{t^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)}=\sum_{n=0}^{\infty}l^{n+1}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)},$$

By substituting from equations (8) and (9) in system (4) we obtain

$$\sum_{n=0}^{\infty}\frac{a^{n+1}}{\Gamma(n\alpha+1)}t^{n\alpha}=b_1-j_1\sum_{n=0}^{\infty}c^{n}t^{n\alpha}-j_2\sum_{n=0}^{\infty}c_1^{n}t^{n\alpha}-f_1\sum_{n=0}^{\infty}\frac{a^{n}}{\Gamma(n\alpha+1)}t^{n\alpha},$$
$$\sum_{n=0}^{\infty}\frac{d^{n+1}}{\Gamma(n\alpha+1)}t^{n\alpha}=j_1\sum_{n=0}^{\infty}c^{n}t^{n\alpha}+j_2\sum_{n=0}^{\infty}c_1^{n}t^{n\alpha}-(b_3+f_1)\sum_{n=0}^{\infty}\frac{d^{n}}{\Gamma(n\alpha+1)}t^{n\alpha},$$
$$\sum_{n=0}^{\infty}\frac{l^{n+1}}{\Gamma(n\alpha+1)}t^{n\alpha}=\frac{j_3b_2}{f_2}\sum_{n=0}^{\infty}\frac{d^{n}}{\Gamma(n\alpha+1)}t^{n\alpha}-j_3\sum_{n=0}^{\infty}c_2^{n}t^{n\alpha}-f_2\sum_{n=0}^{\infty}\frac{l^{n}}{\Gamma(n\alpha+1)}t^{n\alpha},$$

where

$$c^{n}=\sum_{k=0}^{n}\frac{a^{k}d^{n-k}}{\Gamma(k\alpha+1)\Gamma((n-k)\alpha+1)},$$
$$c_1^{n}=\sum_{k=0}^{n}\frac{a^{k}l^{n-k}}{\Gamma(k\alpha+1)\Gamma((n-k)\alpha+1)},$$
$$c_2^{n}=\sum_{k=0}^{n}\frac{l^{k}d^{n-k}}{\Gamma(k\alpha+1)\Gamma((n-k)\alpha+1)}.$$

Then after compilation the summations we find

$$\sum_{n=0}^{\infty}\left(\frac{a^{n+1}}{\Gamma(n\alpha+1)}+j_1c^{n}+j_2c_1^{n}+f_1\frac{a^{n}}{\Gamma(n\alpha+1)}\right)t^{n\alpha}=b_1,$$
$$\sum_{n=0}^{\infty}\left(\frac{d^{n+1}}{\Gamma(n\alpha+1)}-j_1c^{n}-j_2c_1^{n}+(b_3+f_1)\frac{d^{n}}{\Gamma(n\alpha+1)}\right)t^{n\alpha}=0,\qquad(10)$$
$$\sum_{n=0}^{\infty}\left(\frac{l^{n+1}}{\Gamma(n\alpha+1)}+j_3c_2^{n}-\frac{j_3b_2}{f_2}\frac{d^{n}}{\Gamma(n\alpha+1)}+f_2\frac{l^{n}}{\Gamma(n\alpha+1)}\right)t^{n\alpha}=0,$$

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As system (10) non-homogenous then we find the first term by creating the first limit of the summation and get

$$\begin{aligned} a^1 &= b_1 - j_1 a^0 d^0 - j_2 a^0 l^0 - f_1 a^0 \,, \\ d^1 &= j_1 a^0 d^0 + j_2 a^0 l^0 - (b_3 + f_1) d^0, \\ l^1 &= \frac{j_3 b_2}{f_2} d^0 - j_3 d^0 l^0 - f_2 l^0 \,. \end{aligned} \tag{11}$$

Then system (10) become

$$\begin{aligned} &\sum_{n=1}^{\infty} \left(\frac{a^{n+1}}{\Gamma(n\alpha+1)} + j_1 c^n + j_2 c_1^n + f_1 \frac{a^n}{\Gamma(n\alpha+1)} \right) t^{n\alpha} = 0 \,, \\ &\sum_{n=1}^{\infty} \left(\frac{d^{n+1}}{\Gamma(n\alpha+1)} - j_1 c^n - j_2 c_1^n + (b_3 + f_1) \frac{d^n}{\Gamma(n\alpha+1)} \right) t^{n\alpha} = 0 \,, \\ &\sum_{n=1}^{\infty} \left(\frac{l^{n+1}}{\Gamma(n\alpha+1)} + j_3 c_2^n - \frac{j_3 b_2}{f_2} \frac{d^n}{\Gamma(n\alpha+1)} + f_2 \frac{l^n}{\Gamma(n\alpha+1)} \right) t^{n\alpha} = 0 \,, \end{aligned} \tag{12}$$

In system (12), $t^{n\alpha}$ is impossible equal zero then the coefficient that equal zero and we get the recurrence relations which we calculate values of constants $a^n, d^n, l^n,\ n = 1,2,3, \dots, \infty$.

$$\begin{aligned} a^{n+1} &= -j_1 c^n \Gamma(n\alpha+1) - j_2 c_1^n \Gamma(n\alpha+1) - f_1 a^n \,, \\ d^{n+1} &= j_1 c^n \Gamma(n\alpha+1) + j_2 c_1^n \Gamma(n\alpha+1) - (b_3 + f_1) d^n \,, \\ l^{n+1} &= -j_3 c_2^n \Gamma(n\alpha+1) - f_2 l^n + \frac{j_3 b_2}{f_2} d^n \,, \end{aligned} \tag{13}$$

At $n = 1$,

$$\begin{aligned} a^2 &= (-j_1 d^0 - j_2 l_1^0 - f_1)(b_1 - j_1 a^0 d^0 - j_2 a^0 l^0 - f_1 a^0) + j_1 a^0 (j_1 a^0 d^0 + j_2 a^0 l^0 - (b_3 + f_1) d^0) \\ &\quad - j_2 a^0 \left(\frac{j_3 b_2}{f_2} d^0 - j_3 l^0 d^0 - f_2 l^0 \right), \\ d^2 &= (j_1 d^0 + j_2 l^0)(b_1 - j_1 a^0 d^0 - j_2 a^0 l^0 - f_1 a^0) + j_2 a^0 \left(\frac{j_3 b_2}{f_2} d^0 - j_3 l^0 d^0 - f_2 l^0 \right) \\ &\quad + (j_1 a^0 - b_3 - f_1)(j_1 a^0 d^0 + j_2 a^0 l^0 - (b_3 + f_1) d^0), \\ l^2 &= (-j_3 d^0 - f_2) \left(\frac{j_3 b_2}{f_2} d^0 - j_3 l^0 d^0 - f_2 l^0 \right) + \left(\frac{j_3 b_2}{f_2} - j_3 l^0 \right) (j_1 a^0 d^0 + j_2 a^0 l^0 - (b_3 + f_1) d^0) \,. \end{aligned} \tag{14}$$

Similarly we find $a^3, d^3, l^3, a^4, d^4, l^4, \dots \dots$

Substituting from (11), (14) into (8), we have the solution in the infinite series form at the follow :

$$\begin{aligned} x(t) &= a^0 + a^1 \frac{t^\alpha}{\Gamma(\alpha+1)} + a^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + a^3 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots, \\ y(t) &= d^0 + d^1 \frac{t^\alpha}{\Gamma(\alpha+1)} + d^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + d^3 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots, \\ z(t) &= l^0 + l^1 \frac{t^\alpha}{\Gamma(\alpha+1)} + l^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + l^3 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots. \end{aligned}$$

The numerical results that we obtained by solving the system (4) by GMLFM showing in the following diagram. Such that in figure 1, we comparing the results from RK4 method and that obtained by GMLFM. In figure 2, we make it clear the different values of α for our solution by GMLFM.

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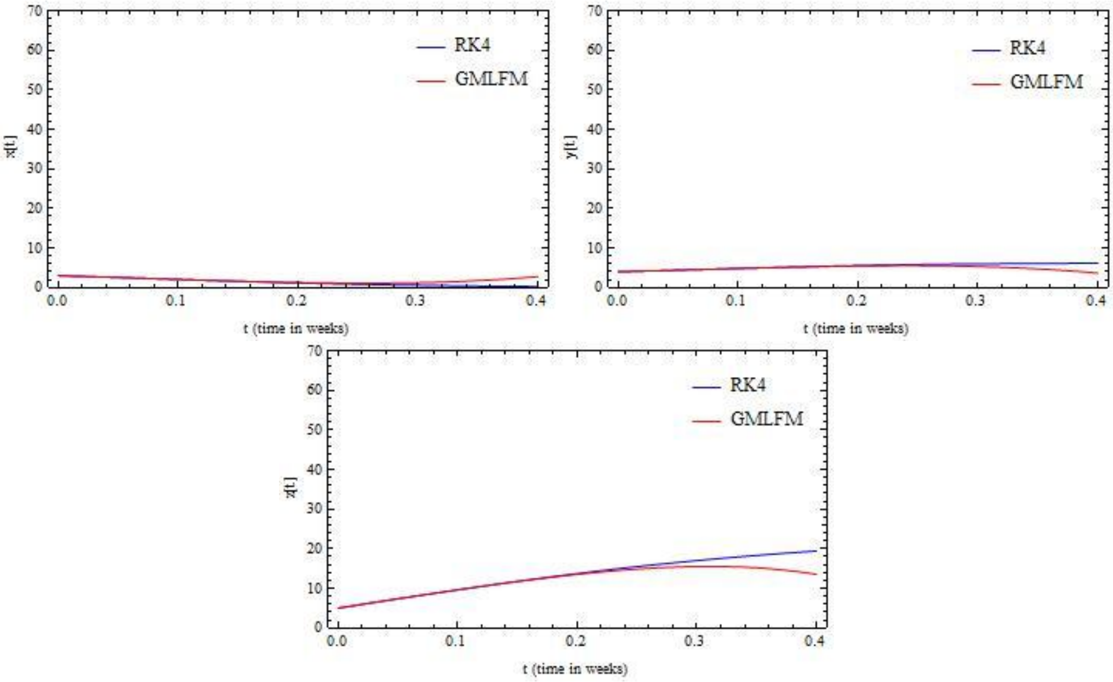

Figure 1: The comparison plots x, y and z for $\alpha = 1$ using "GMLFM" and "RK4" method.

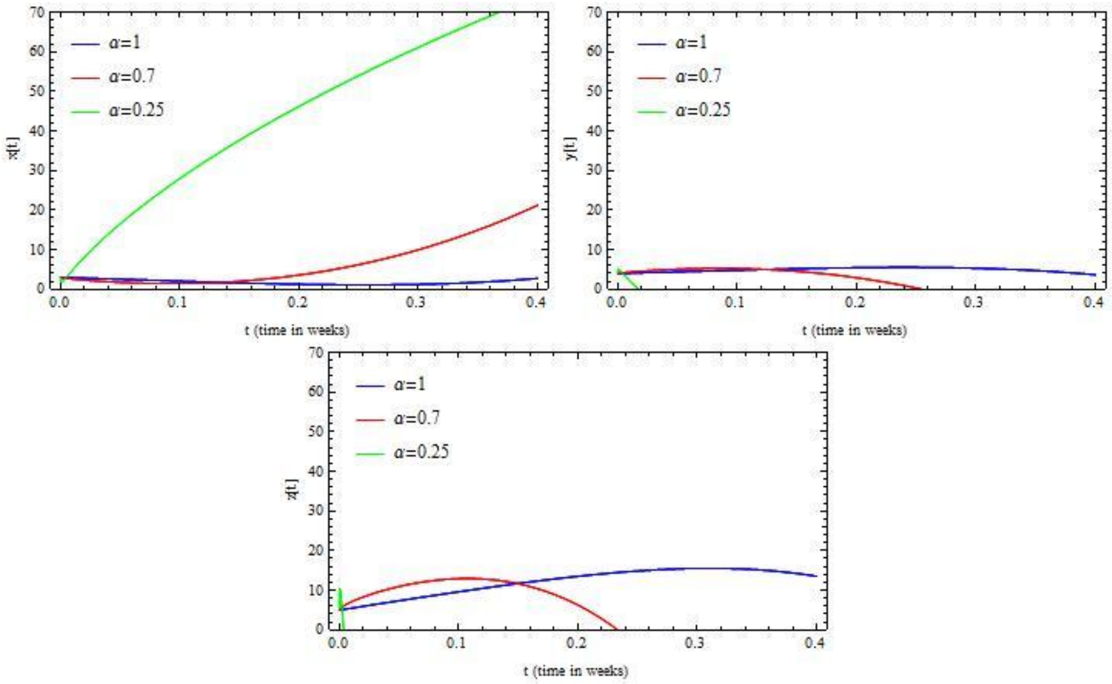

Figure 2: Plots show that x, y and z with different values of α.

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V. CONCLUSION

In this paper, epidemic model of a vector-born disease is solved by GMLFM and the obtained analytical solutions are presented in the form of infinite series. The flexibility of GMLFM in solving this model of fractional order make it the more suitable method to apply. The final results of GMLFM were compared by the results from RK4 method and we find that close harmony between two methods which make GMLFM is the easiest and strongest.

REFERENCES

[1] Arafa. A. A. M, Rida. S. Z, Khalil. M, "Fractional modeling dynamics of HIV and $CD4^+$ T-cells during primary infection", Nonlinear Biomedical Physics, vol: 6, 2012, pg no:01-07.
[2] Arafa. A. A. M, Rida. S. Z, Ali. H. M, "Generalized Mittag-Leffler function method for solving Lorenz system", International Journal of Innovation and Applied Studies, vol: 3, 2013, pg no: 105-111.
[3] Arafa. A. A. M, Rida. S. Z, Mohammadein. A. A, Ali. H. M, "Solving Nonlinear Fractional Differential Equation by Generalized Mittag-Leffler Function Method", Commun. Theor. Phys. journal, vol:59, 2013, pg no: 661-663.
[4] Arqub. O. A, El-Ajou. A, "Solution of the fractional epidemic model by homotopy analysis method", J. King Saud Uni. Sci., vol: 25, 2013, pg no: 73-81.
[5] Awawdeh. F, Adawi. A, Mustafa. Z, "Solutions of the SIR models of epidemics using HAM", Chaos,Solitons and Fractals, vol: 42, 2009, pg no: 3047-3052.
[6] Benchohra. M, Graef. J. R, Hamani. S, "Existence results for boundary value problems with nonlinear fractional differential equations", Appl. Anal., vol: 87, 2008, pg no: 851-863.
[7] Haq. F, Shah. K, Khan. A, Shahzad. M, Rahman. G, "Numerical Solution of Fractional Order Epidemic Model of a Vector Born Disease by Laplace Adomian Decomposition Method", Journal of Mathematics, vol: 49, 2017, pg no: 13-22.
[8] Podlubny. I," Fractional Differential Equations", Academic Press, London, 1999.
[9] Rafei. M, Daniali. H, Ganji. D. D, "Variational iteration method for solving the epidemic model and the prey and predator problem", Appl. Math. Comput., vol: 186, 2007, pg no: 1701-1709.
[10] Rafei. M, Ganji. D. D, Daniali. H, "Solution of the epidemic model by homotopy perturbation method", Appl. Math. Comput., vol: 187, 2007, pg no: 1056-1062.
[11] Rida. S. Z, Abdel Rady. A. S, Arafa. A. A. M, Khalil. M, "Approximate analytical solution of the fractional epidemic model", IJMR, vol: 1, 2012, pg no: 17-19.
[12] Rida. S. Z, Arafa. A. A. M, "New Method for Solving Linear Fractional Differential Equations", International Journal of Differential Equations (2011), Article ID 814132, 8 pages, doi:10.1155/2011/814132.
[13] Rida. S. Z, Arafa. A. A. M, Gaber. Y. A, "Solution of the fractional epidemic model by L-ADM", Frac. Calc. Appl., vol: 7, 2016, pg no: 189-195.
[14] Srivastava. P. K, Chandra. P, "Modeling the dynamics of HIV and $CD4^+$ T-cells during primary infection", Nonlinear Anal. Real World Appl., vol: 11, 2010, pg no: 612-618.
[15] Thieme. H. R, "Convergence results and a Poincar Bendixson trichotomy for asymptotically autonomous differential equations", J. Math. Biol., vol: 30, 1992, pg no: 755-763.
[16] Wei. H. M, Li. X. Z, Martcheva. M, "An epidemic model of a vector-born disease with direct transmission and time delay", Math. Anal. appl., vol: 342, 2008, pg no: 895-908.
[17] Zurigat. M, Momani. S, Alawneh. A, "Analytical approximate solutions of systems of fractional algebraic differential equations by homotopy analysis method", Computers and Mathematics with Applications, vol: 59, 2010, pg no: 1227-1235.
[18] Zurigat. M, Momani. S, Odibat. Z, Alawneh. A, "The homotopy analysis method for handling systems of fractional differential equations", Applied Mathematical Modelling, vol: 34, 2010, pg no: 24-35.