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Coefficient Estimates of Bi-Univalent Functions Based on Subordination Involving Srivastava-Attiya Operator

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ABSTRACT: The purpose of the present paper is to introduce and investigate two new subclasses of bi-univalent function of complex order defined in the open unit disk, which are associated with Srivastava-Attiya operator and satisfying subordinate conditions. Furthermore, we find estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in the new subclasses. Several (known or new) consequences of the results are also pointed out.

KEYWORDS: Analytic function; univalent function; bi-univalent function; bi-starlike and bi-convex function; subordination; Srivastava-Attiya operator.

I.INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1)$$

which are analytic in the open unit disk $U = \{z: |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Further, let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in U. Some of the important and well-investigated subclasses of the class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*(\alpha)$ $(0 \le \alpha < 1)$ of starlike functions of order α in U and the class $\mathcal{K}(\alpha)(0 \le \alpha < 1)$ of convex functions of order α in U.

The Koebe One-Quarter Theorem [5] states that the image of U under every function f from $\mathcal{S}$ contains a disk of radius 1/4. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}\big(f(z)\big) = z \quad (z \in U)\text{and} \quad f(f^{-1}(w)) = w \quad \left(|w| < r_0(f);\ r_0(f) \ge \frac{1}{4}\right),$$

where

$$f^{-1}(w) = g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \qquad (2)$$

A function $f \in \mathcal{A}$is said tobebi-univalent in U if both f and f^{-1} are univalent in U. We denote by Σ the class of all bi-univalent functions in U given by the Taylor-Maclaurin series expansion (1).

If f and g are analytic functions in U, we say that f is subordinate to g, written $f(z) \prec g(z)$ if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g\big(w(z)\big)$, $z \in U$. Ma and Minda [11] unified various subclasses of starlike and convex functions for which either of the quantity $\frac{zf'(z)}{f(z)}$ (or) $1 + \frac{zf''(z)}{f'(z)}$ issubordinate to a more general superordinate function. For this purpose, they considered an analytic function ϕ with positive real part in the open unit disk $U, \phi(0) = 1, \phi'(0) > 0$, and ϕ maps U onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike functions consists of functions $f \in \mathcal{A}$ satisfying the subordination $\frac{zf'(z)}{f(z)} \prec \phi(z)$.Similary, the class of Ma-Minda convex functions of functions $f \in \mathcal{A}$satisfying the subordination $1 + \frac{zf''(z)}{f'(z)} \prec \phi(z)$. A function f is bi- starlike of Ma-Minda type or bi-convex of Ma-Minda type if both f and f^{-1} are respectively Ma-Minda starlike or convex. These classes are denoted respectively by $\mathcal{S}_\Sigma^*(\phi)$ and $\mathcal{K}_\Sigma(\phi)$. In the sequel, it is assumed that ϕ is an analytic function with positive real part in the unit disk U, satisfying $\phi(0)=1, \phi'(0)>0$, and $\phi(U)$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\phi(z)=1+B_1 z+B_2 z^2+B_3 z^3+\cdots, \quad (B_1>0). \tag{3}$$

For two function $f(z)\in\mathcal{A}$ given by (1) and $g(z)\in\mathcal{A}$ given by $g(z)=z+\sum_{n=2}^{\infty} b_n z^n$, the Hadamard product (or convolution) of f and g is defined by

$$(f*g)(z)=z+\sum_{n=2}^{\infty} a_n b_n z^n=(g*f)(z). \tag{4}$$

TheSrivastava-Attiya convolution operator [14], $\mathcal{J}_b^s f(z)$ is defined in terms of the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ as follows:

$$\Phi(z,s,a)=\sum_{n=0}^{\infty}\frac{z^k}{(n+a)^s} \tag{5}$$

$(a\in\mathbb{C}\setminus\mathbb{Z}_0^-; s\in\mathbb{C}\ when\ |z|<1; Re\ (s)>1\ and\ |z|=1)$,
where $\mathbb{Z}_0^-=\mathbb{Z}\backslash\mathbb{N}, (\mathbb{Z}=\{0,\pm1,\pm2,\ldots\};\ \mathbb{N}=\{1,2,3,\ldots\})$.

Properties of the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ can be found in the works of Choi and Srivastava [3], Luo and Srivastava [10], Gary et al. [7]. Srivastava and Attiya [14] have introduced the linear operator $\mathcal{J}_b^s:\mathcal{A}\to\mathcal{A}$ defined by the Hadamard product as follows:

$$\mathcal{J}_b^s f(z)=G_{s,b}*f(z)(z\in U; b\in\mathbb{C}\backslash\mathbb{Z}_0^-; s\in\mathbb{C}; f\in\mathcal{A}), \tag{6}$$

where $G_{s,b}(z)=(1+b)^s[\Phi(z,s,b)-b^{-s}](z\in U)$.
Using equations (1), (5) and (6), we have$\mathcal{J}_b^s f(z)=z+\sum_{n=2}^{\infty}\Gamma_n a_n z^n$, where

$$\Gamma_n=\left|\left(\frac{1+b}{n+b}\right)^s\right| \qquad ,(s\in\mathbb{C}; b\in\mathbb{C}\backslash\{0,-1,-2,\ldots\}. \tag{7}$$

For $f(z)\in\mathcal{A}$ and $z\in U$, Srivastava and Attiya in [14] showed that

$$\mathcal{J}_b^0 f(z)=f(z), \mathcal{J}_0^1 f(z)=\int_0^z\frac{f(t)}{t}dt\ =A\,f(z),$$

$$\mathcal{J}_\gamma^1 f(z)=\frac{1+\gamma}{z^\gamma}\int_0^z t^{\gamma-1}f(t)\,dt\ =J_\gamma f(z)(\gamma>-1), \qquad \mathcal{J}_1^\sigma f(z)=z+\sum_{n=2}^{\infty}\left(\frac{2}{n+1}\right)^\sigma a_n z^n=I^\sigma f(z)(\sigma>0),$$

where $Af(z)$ and $J_\gamma f(z)$ are the integral operators introduced by Alexander [1] and Bernardi [2], respectively, and $I^\sigma f(z)$ is the Jung-Kim-Srivastava integral operator [8] closely related to some multiplier transformation studied by Flett [6].

Recently, a study on bi-univalent function class Σ has increased. A number of articles discussing on non-sharp coefficient estimates for the first two coefficient$|a_2|$ and $|a_3|$ of (1). But the coefficient problem for each of the following Taylor- Maclaurin coefficients:

$$|a_n| \qquad (n\in\mathbb{N}\backslash\{1,2\};\ \mathbb{N}=\{1,2,3,\ldots\}$$

is still an open problem (see [15]). Many researchers (see [4,9,12,13,15]) have recently introduced and investigated several interesting subclasses of the bi-univalent function class Σ and they have found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

Motivated by the earlier work of Deniz [4], in the present paper, we introduce two new subclasses of the function class Σ of complex order $\tau\in\mathbb{C}\backslash\{0\}$, involving Srivastava-Attiya operator and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the new subclasses of function class Σ. Several related classes are also considered, and connections to earlier known results are made.

Definition 1. A function $f\in\Sigma$ given by (1) is said to be in the class $\mathcal{H}_\Sigma^{s,b}(\tau,\ \lambda;\phi)$ if the following conditions are satisfied:

$$1+\frac{1}{\tau}\left(\frac{z\left[\lambda z\left(\mathcal{J}_b^s f(z)\right)'+(1-\lambda)\mathcal{J}_b^s f(z)\right]'}{\lambda z\left(\mathcal{J}_b^s f(z)\right)'+(1-\lambda)\mathcal{J}_b^s f(z)}-1\right)\prec\phi(z) \tag{8}$$

and

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$$1+\frac{1}{\tau}\left(\frac{w\left[\lambda w\left(\mathcal{J}_b^s g(w)\right)^{'}+(1-\lambda)\mathcal{J}_b^s g(w)\right]^{'}}{\lambda w\left(\mathcal{J}_b^s g(w)\right)^{'}+(1-\lambda)\mathcal{J}_b^s g(w)}-1\right)\prec\phi(w), \tag{9}$$

where $\tau\in\mathbb{C}\backslash\{0\}; 0\leq\lambda\leq1; z,w\in U$ and the function g is given by (2).

Specializing the parameters b,s and λ suitably, several known and new subclasses can be obtained from the class $\mathcal{H}_\Sigma^{s,b}(\tau,\lambda;\phi)$. We present some of the subclasses of $\mathcal{H}_\Sigma^{s,b}(\tau,\lambda;\phi)$, as given below:

Example 1. For $\lambda=0$ and $\tau\in\mathbb{C}\backslash\{0\}$, a function $f\in\Sigma$, given by (1) is said to be in the class $\mathcal{S}_\Sigma^{s,b}(\tau;\phi)$ if the following conditions are satisfied:

$$1+\frac{1}{\tau}\left(\frac{z\left(\mathcal{J}_b^s f(z)\right)^{'}}{\mathcal{J}_b^s f(z)}-1\right)\prec\phi(z)(z\in U)\text{and}\qquad 1+\frac{1}{\tau}\left(\frac{w\left(\mathcal{J}_b^s g(w)\right)^{'}}{\mathcal{J}_b^s g(w)}-1\right)\prec\phi(w)(w\in U),$$

where the function g is given by (2).

Example 2. For $\lambda=1$ and $\tau\in\mathbb{C}\backslash\{0\}$, a function $f\in\Sigma$, given by (1) is said to be in the class $\mathcal{K}_\Sigma^{s,b}(\tau;\phi)$ if the following conditions are satisfied:

$$1+\frac{1}{\tau}\left(\frac{z\left(\mathcal{J}_b^s f(z)\right)^{''}}{\left(\mathcal{J}_b^s f(z)\right)^{'}}\right)\prec\phi(z)(z\in \mathrm{U})\qquad\text{and}\qquad 1+\frac{1}{\tau}\left(\frac{w\left(\mathcal{J}_b^s g(w)\right)^{''}}{\left(\mathcal{J}_b^s g(w)\right)^{'}}\right)\prec\phi(w)(w\in U),$$

where the function g is given by (2).

In particular, for $s=0$, we note that $\mathcal{J}_b^s f(z)=f(z)$ for all $f\in\mathcal{A}$, and thus, the class $\mathcal{H}_\Sigma^{s,b}(\tau;\lambda;\phi)$ reduces to the following subclasses:

Example 3. For $s=0$, a function $f\in\Sigma$, given by (1) is said to be in the class $\mathcal{H}_\Sigma(\tau,\lambda;\phi)$ if the following conditions are satisfied:

$$1+\frac{1}{\tau}\left(\frac{z[\lambda z f^{'}(z)+(1-\lambda)f(z)]^{'}}{\lambda z f^{'}(z)+(1-\lambda)f(z)}-1\right)\prec\phi(z)\quad\text{and}\quad 1+\frac{1}{\tau}\left(\frac{w[\lambda w g^{'}(w)+(1-\lambda)g(w)]^{'}}{\lambda w g^{'}(w)+(1-\lambda)g(w)}-1\right)\prec\phi(w),$$

where $\tau\in\mathbb{C}\backslash\{0\}; 0\leq\lambda\leq1; z,w\in U$ and the function g is given by (2).

Example 4. For $s=0$ and $\lambda=0$, a function $f\in\Sigma$, given by (1) is said to be in the class $\mathcal{S}_\Sigma^{*}(\tau;\phi)$ if the following conditions are satisfied:

$$1+\frac{1}{\tau}\left(\frac{zf^{'}(z)}{f(z)}-1\right)\prec\phi(z)(z\in U)\text{and}\qquad 1+\frac{1}{\tau}\left(\frac{wg^{'}(w)}{g(w)}-1\right)\prec\phi(w)(w\in U),$$

where $\tau\in\mathbb{C}\backslash\{0\}$and the function g is given by (2).

Example 5. For $s=0$ and $\lambda=1$, a function $f\in\Sigma$, given by (1) is said to be in the class $\mathcal{K}_\Sigma(\tau;\phi)$ if the following conditions are satisfied:

$$1+\frac{1}{\tau}\left(\frac{zf^{''}(z)}{f^{'}(z)}\right)\prec\phi(z)(z\in U)\qquad\text{and}\qquad 1+\frac{1}{\tau}\left(\frac{wg^{''}(w)}{g^{'}(w)}\right)\prec\phi(w)(w\in U),$$

where $\tau\in\mathbb{C}\backslash\{0\}$and the function g is given by (2).

Definition 2. A function $f\in\Sigma$, given by (1) is said to be in the class $\mathcal{N}_\Sigma^{s,b}(\tau,\ \lambda;\phi)$ if the following conditions are satisfied:

$$1+\frac{1}{\tau}\left[\frac{z\left(\mathcal{J}_b^s f(z)\right)^{'}+z^2\left(\mathcal{J}_b^s f(z)\right)^{''}}{(1-\lambda)z+\lambda z\left(\mathcal{J}_b^s f(z)\right)^{'}}-1\right]\prec\phi(z) \tag{10}$$

and

$$1+\frac{1}{\tau}\left[\frac{w\left(\mathcal{J}_b^s g(w)\right)^{'}+w^2\left(\mathcal{J}_b^s g(w)\right)^{''}}{(1-\lambda)w+\lambda w\left(\mathcal{J}_b^s g(w)\right)^{'}}-1\right]\prec\phi(w), \tag{11}$$

where $\tau\in\mathbb{C}\backslash\{0\}; 0\leq\lambda\leq1; z,w\in U$ and the function g is given by (2).

On specializing the parameters b,s and λ suitably, several known and new subclasses can be obtained from the class $\mathcal{N}_\Sigma^{s,b}(\tau,\lambda;\phi)$. We present some of the subclasses of $\mathcal{N}_\Sigma^{s,b}(\tau,\lambda;\phi)$, as given below:

Example 6. For $\lambda=0$, a function $f\in\Sigma$, given by (1) is said to be in the class $\mathcal{G}_\Sigma^{s,b}(\tau;\phi)$ if the following conditions are satisfied:

$$1+\frac{1}{\tau}\left[\left(\mathcal{J}_b^s f(z)\right)^{'}+z\left(\mathcal{J}_b^s f(z)\right)^{''}-1\right]\prec\phi(z)(z\in U)\text{and } 1+\frac{1}{\tau}\left[\left(\mathcal{J}_b^s g(w)\right)^{'}+w\left(\mathcal{J}_b^s g(w)\right)^{''}-1\right]\prec\phi(w)(w\in U),$$

where $\tau\in\mathbb{C}\backslash\{0\}$ and the function g is given by (2).

Remark 1. We note that by taking $\lambda=1$, the class $\mathcal{N}_{\Sigma}^{s,b}(\tau,\lambda;\phi)$ reduce to the class $\mathcal{K}_{\Sigma}^{s,b}(\tau;\phi)$ which given in example (2).

Example 7. For $s=0$, a function $f\in\Sigma$, given by (1) is said to be in the class $\mathcal{N}_{\Sigma}(\tau,\ \lambda;\ \phi)$ if the following conditions are satisfied:

$$1+\frac{1}{\tau}\left[\frac{z\,f^{'}(z)+z^2f^{''}(z)}{(1-\lambda)z+\lambda z\,f^{'}(z)}-1\right]\prec\phi(z)\quad\text{and}\quad 1+\frac{1}{\tau}\left[\frac{w\,g^{'}(w)+w^2g^{''}(w)}{(1-\lambda)w+\lambda w\,g^{'}(w)}-1\right]\prec\phi(w),$$

where $\tau\in\mathbb{C}\backslash\{0\};0\le\lambda\le1;z,w\in U$ and the function g is given by (2).

Example 8. If $s=0$ and $\lambda=0$, a function $f\in\Sigma$, given by (1) is said to be in the class $\mathcal{G}_{\Sigma}(\tau;\phi)$ if the following conditions are satisfied:

$$1+\frac{1}{\tau}(f^{'}(z)+zf^{''}(z)-1)\prec\phi(z)(z\in U)\text{and}\quad 1+\frac{1}{\tau}(g^{'}(w)+wg^{''}(w)-1)\prec\phi(w)(w\in U),$$

where $\tau\in\mathbb{C}\backslash\{0\}$and the function g is given by (2).

Remark 2. We note that by taking $s=0$ and $\lambda=1$, the class $\mathcal{N}_{\Sigma}^{s,b}(\tau,\lambda;\phi)$ reduce to the class $\mathcal{K}_{\Sigma}(\tau;\phi)$ which given in example (5).

In order to derive our main results, we have to recall here the following lemma[5].

Lemma 1. If $h\in\mathcal{P}$, then $|b_k|\le2$ for each $k\in\mathbb{N}$, where $\mathcal{P}$ is the family of all functions h, analytic in U, for which

$$Re\big(h(z)\big)>0\qquad(z\in U)\text{where } h(z)=1+b_1z+b_2z^2+\cdots\qquad(z\in U)$$

II. COEFFICIENT ESTIMATES FOR THE FUNCTIONCLASS $\mathcal{H}_{\Sigma}^{s,b}(\tau,\lambda;\phi)AND\mathcal{N}_{\Sigma}^{s,b}(\tau,\lambda;\phi)$

Theorem 1. Let the function $f(z)$ given by (1) be in the class $\mathcal{H}_{\Sigma}^{s,b}(\tau,\lambda;\phi)$. Then

$$|a_2|\le\frac{|\tau|B_1\sqrt{B_1}}{\sqrt{|(1+\lambda)^2[(B_1-B_2)-\tau B_1^2]\Gamma_2^2+2\tau(1+2\lambda)B_1^2\Gamma_3|}}\tag{12}$$

and

$$|a_3|\le\frac{|\tau|^2B_1^2}{(1+\lambda)^2\Gamma_2^2}+\frac{|\tau|B_1}{2(1+2\lambda)\Gamma_3}.\tag{13}$$

Proof. It follows from (8) and (9) that

$$1+\frac{1}{\tau}\left(\frac{z\left[\lambda z\left(\mathcal{J}_b^s f(z)\right)^{'}+(1-\lambda)\mathcal{J}_b^s f(z)\right]^{'}}{\lambda z\left(\mathcal{J}_b^s f(z)\right)^{'}+(1-\lambda)\mathcal{J}_b^s f(z)}-1\right)=\phi\big(u(z)\big)\tag{14}$$

and

$$1+\frac{1}{\tau}\left(\frac{w\left[\lambda w\left(\mathcal{J}_b^s g(w)\right)^{'}+(1-\lambda)\mathcal{J}_b^s g(w)\right]^{'}}{\lambda w\left(\mathcal{J}_b^s g(w)\right)^{'}+(1-\lambda)\mathcal{J}_b^s g(w)}-1\right)=\phi\big(v(w)\big).\tag{15}$$

Define the function $p(z)$ and $q(z)$ by

$$p(z)=\frac{1+u(z)}{1-u(z)}=1+p_1z+p_2z+\cdots\quad\text{and}\quad q(z)=\frac{1+v(z)}{1-v(z)}=1+q_1z+q_2z+\cdots,$$

or equivalently,

$$u(z)=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}\left[p_1z+\left(p_2-\frac{p_1^2}{2}\right)z^2+\cdots\right]\text{and } v(z)=\frac{q(z)-1}{q(z)+1}=\frac{1}{2}\left[q_1z+\left(q_2-\frac{q_1^2}{2}\right)z^2+\cdots\right].$$

Then $p(z)$ and $q(z)$ are analytic in U with $p(0)=1=q(0)$. Since $u,v:U\to U$, the functions $p(z)$ and $q(z)$ have a positive real part in U, and $|p_i|\le2$ and $|q_i|\le2$ for each i.

Since $p(z)$ and $q(z)$ in $\mathcal{P}$, we have the following forms:

$$\phi\big(u(z)\big)=\phi\left(\frac{1}{2}\left[p_1z+\left(p_2-\frac{p_1^2}{2}\right)z^2+\cdots\right]\right)=\frac{1}{2}B_1p_1z+\left[\frac{1}{2}B_1\left(p_2-\frac{p_1^2}{2}\right)+\frac{1}{4}B_2\right]z^2+\cdots\tag{16}$$

and

$$\phi(v(w)) = \phi\left(\frac{1}{2}\left[q_1 w + \left(q_2 - \frac{q_1^2}{2}\right)w^2 + \cdots\right]\right) = \frac{1}{2}B_1 q_1 w + \left[\frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2\right]w^2 + \cdots. \quad (17)$$

Now, equating the coefficients in (14) and (15), we get

$$\frac{1}{\tau}(1+\lambda)\Gamma_2 a_2 = \frac{1}{2}B_1 p_1, \quad (18)$$

$$\frac{1}{\tau}[2(1+2\lambda)\Gamma_3 a_3 - (1+\lambda)^2\Gamma_2^2 a_2^2] = \frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2 p_1^2, \quad (19)$$

$$-\frac{1}{\tau}(1+\lambda)\Gamma_2 a_2 = \frac{1}{2}B_1 q_1 \quad (20)$$

and

$$\frac{1}{\tau}[2(1+2\lambda)\Gamma_3(2a_2^2 - a_3) - (1+\lambda)^2\Gamma_2^2 a_2^2] = \frac{1}{2}B_1\left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4}B_2 q_1^2. \quad (21)$$

From (18) and (20), we find that

$$a_2 = \frac{\tau B_1 p_1}{2(1+\lambda)\Gamma_2} = \frac{-\tau B_1 q_1}{2(1+\lambda)\Gamma_2}, \quad (22)$$

which implies

$$p_1 = -q_1 \quad (23)$$

and

$$8(1+\lambda)^2\Gamma_2^2 a_2^2 = \tau^2 B_1^2(p_1^2 + q_1^2) \quad (24)$$

Adding (19) and (21), by using (22) and (23), we obtain

$$4((1+\lambda)^2[(B_1 - B_2) - \tau B_1^2]\Gamma_2^2 + 2\tau(1+2\lambda)B_1^2\Gamma_3)a_2^2 = \tau^2 B_1^3(p_2 + q_2). \quad (25)$$

Thus,

$$a_2^2 = \frac{\tau^2 B_1^3(p_2 + q_2)}{4((1+\lambda)^2[(B_1 - B_2) - \tau B_1^2]\Gamma_2^2 + 2\tau(1+2\lambda)B_1^2\Gamma_3)}. \quad (26)$$

Applying Lemma (1) for the coefficients p_2 and q_2, we immediately have

$$|a_2|^2 \le \frac{|\tau|^2 B_1^3}{|(1+\lambda)^2[(B_1 - B_2) - \tau B_1^2]\Gamma_2^2 + 2\tau(1+2\lambda)B_1^2\Gamma_3|}. \quad (27)$$

Hence,

$$|a_2| \le \frac{|\tau| B_1\sqrt{B_1}}{\sqrt{|(1+\lambda)^2[(B_1 - B_2) - \tau B_1^2]\Gamma_2^2 + 2\tau(1+2\lambda)B_1^2\Gamma_3|}}. \quad (28)$$

This gives the bound on $|a_2|$ as asserted in (12).

Next, in order to find the bound on $|a_3|$, by subtracting (21) from (19), we get

$$\frac{4}{\tau}(1+2\lambda)\,(a_3 - a_2^2)\,\Gamma_3 = \frac{B_1}{2}(p_2 - q_2) + \frac{(B_2 - B_1)}{4}(p_1^2 - q_1^2). \quad (29)$$

It follows from (22), (23) and (29) that

$$a_3 = \frac{\tau^2 B_1^2(p_1^2 + q_1^2)}{8(1+\lambda)^2\Gamma_2^2} + \frac{\tau B_1(p_2 - q_2)}{8(1+2\lambda)\Gamma_3}.$$

Applying Lemma (1) once again for the coefficients p_2 and q_2, we readily get

$$|a_3| \le \frac{|\tau|^2 B_1^2}{(1+\lambda)^2\Gamma_2^2} + \frac{|\tau| B_1}{2(1+2\lambda)\Gamma_3}.$$

This completes the proof of Theorem (1).

By putting $\lambda = 0$ in Theorem (1), we have the following corollary.

Corollary 1. Let the function $f(z)$ given by (1) be in the class $\mathcal{S}_\Sigma^{s,b}(\tau;\phi)$. Then

$$|a_2| \le \frac{|\tau| B_1\sqrt{B_1}}{\sqrt{|[(B_1 - B_2) - \tau B_1^2]\Gamma_2^2 + 2\tau B_1^2\Gamma_3|}} \text{ and } |a_3| \le \frac{|\tau|^2 B_1^2}{\Gamma_2^2} + \frac{|\tau| B_1}{2\Gamma_3}.$$

By putting $\lambda = 1$ in Theorem (1), we have the following corollary.

Corollary 2. Let the function $f(z)$ given by (1) be in the class $\mathcal{K}_\Sigma^{s,b}(\tau;\phi)$. Then

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$$|a_2| \le \frac{|\tau|B_1\sqrt{B_1}}{\sqrt{|4[(B_1-B_2)-\tau B_1^2]\Gamma_2^2+6\tau B_1^2\Gamma_3|}} \text{and} |a_3| \le \frac{|\tau|^2B_1^2}{4\Gamma_2^2}+\frac{|\tau|B_1}{6\Gamma_3}.$$

Taking $s=0$, we have $\Gamma_n=1$ $(n\ge 2)$ in Theorem (1), and we can state the coefficient estimates for the functions in the subclass $\mathcal{H}_\Sigma(\tau,\lambda;\phi)$.

Corollary 3. Let the function $f(z)$ given by (1) be in the class $\mathcal{H}_\Sigma(\tau,\lambda;\ \phi)$.Then

$$|a_2| \le \frac{|\tau|B_1\sqrt{B_1}}{\sqrt{|(1+\lambda)^2(B_1-B_2)+\tau(1+2\lambda-\lambda^2)B_1^2|}} \text{ and} |a_3| \le \frac{|\tau|^2B_1^2}{(1+\lambda)^2}+\frac{|\tau|B_1}{2(1+2\lambda)}.$$

Taking $\lambda=1$ in Corollary 3, we get the following corollary

Corollary 4. Let the function $f(z)$ given by (1) be in the class $\mathcal{K}_\Sigma(\tau;\phi)$.Then

$$|a_2| \le \frac{|\tau|B_1\sqrt{B_1}}{\sqrt{|4(B_1-B_2)+2\tau B_1^2|}} \text{and} |a_3| \le \frac{|\tau|^2B_1^2}{4}+\frac{|\tau|B_1}{6}.$$

Remark 3. Putting $\lambda=0$ in Corollary (3), we obtain the corresponding result given by Murugusundaramoorthy et al. [12].

Theorem 2. Let the function $f(z)$ given by (1) be in the class $\mathcal{N}_\Sigma^{s,b}(\tau,\lambda;\phi)$.Then

$$|a_2| \le \frac{|\tau|B_1\sqrt{B_1}}{\sqrt{|4[\tau(\lambda^2-2\lambda)B_1^2+(2-\lambda)^2(B_1-B_2)]\Gamma_2^2+3\tau(3-\lambda)B_1^2\Gamma_3|}} \tag{30}$$

and

$$|a_3| \le \frac{|\tau|^2B_1^2}{4(2-\lambda)^2\Gamma_2^2}+\frac{|\tau|B_1}{3(3-\lambda)\Gamma_3}. \tag{31}$$

Proof. We can write the argument inequalities in (10) and (11) equivalently as follows:

$$1+\frac{1}{\tau}\left(\frac{z\left(\mathcal{J}_b^s f(z)\right)'+z^2\left(\mathcal{J}_b^s f(z)\right)''}{(1-\lambda)z+\lambda z\left(\mathcal{J}_b^s f(z)\right)'}-1\right)=\phi\left(u(z)\right) \tag{32}$$

and

$$1+\frac{1}{\tau}\left(\frac{w\left(\mathcal{J}_b^s g(w)\right)'+w^2\left(\mathcal{J}_b^s g(w)\right)''}{(1-\lambda)w+\lambda w\left(\mathcal{J}_b^s g(w)\right)'}-1\right)=\phi\left(v(w)\right) \tag{33}$$

and proceeding as in the proof of Theorem (1), from (32) and (33), we can arrive the following relations

$$\frac{2}{\tau}(2-\lambda)\Gamma_2a_2=\frac{1}{2}B_1p_1, \tag{34}$$

$$\frac{1}{\tau}[4(\lambda^2-2\lambda)\Gamma_2^2a_2^2+3(3-\lambda)\Gamma_3a_3]=\frac{1}{2}B_1\left(p_2-\frac{p_1^2}{2}\right)+\frac{1}{4}B_2p_1^2, \tag{35}$$

$$\frac{-2}{\tau}(2-\lambda)\Gamma_2a_2=\frac{1}{2}B_1q_1 \tag{36}$$

and

$$\frac{1}{\tau}[4(\lambda^2-2\lambda)\Gamma_2^2a_2^2+3(3-\lambda)\Gamma_3(2a_2^2-a_3)]=\frac{1}{2}B_1\left(q_2-\frac{q_1^2}{2}\right)+\frac{1}{4}B_2q_1^2. \tag{37}$$

From (34) and (36), we find that

$$a_2=\frac{\tau B_1p_1}{4(2-\lambda)\Gamma_2}=\frac{-\tau B_1q_1}{4(2-\lambda)\Gamma_2}, \tag{38}$$

which implies

$$p_1=-q_1, \tag{39}$$

and

$$32(2-\lambda)^2\Gamma_2^2a_2^2=\tau^2B_1^2(p_1^2+q_1^2). \tag{40}$$

Adding (35) and (37), by using (38) and (39), we obtain

$$4(4[\tau(\lambda^2-2\lambda)B_1^2+(2-\lambda)^2(B_1-B_2)]\Gamma_2^2+3\tau(3-\lambda)B_1^2\Gamma_3)a_2^2=\tau^2B_1^3(p_2+q_2). \tag{41}$$

Thus,

$$a_2^2=\frac{\tau^2B_1^3(p_2+q_2)}{4(4[\tau(\lambda^2-2\lambda)B_1^2+(2-\lambda)^2(B_1-B_2)]\Gamma_2^2+3\tau(3-\lambda)B_1^2\Gamma_3)}. \tag{42}$$

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Applying Lemma (1) for the coefficients p_2 and q_2, we immediately have

$$|a_2|^2 \le \frac{|\tau|^2 B_1^3}{|4[\tau(\lambda^2-2\lambda)B_1^2+(2-\lambda)^2(B_1-B_2)]\Gamma_2^2+3\tau(3-\lambda)B_1^2\Gamma_3|}. \quad (43)$$

Hence,

$$|a_2| \le \frac{|\tau|B_1\sqrt{B_1}}{\sqrt{|4[\tau(\lambda^2-2\lambda)B_1^2+(2-\lambda)^2(B_1-B_2)]\Gamma_2^2+3\tau(3-\lambda)B_1^2\Gamma_3|}}. \quad (44)$$

This gives the bound on $|a_2|$ as asserted in (30).

Next, in order to find the bound on $|a_3|$, by subtracting (37) from (35), we get

$$\frac{6(3-\lambda)}{\tau}\Gamma_3 a_3 - \frac{6(3-\lambda)}{\tau}\Gamma_3 a_2^2 = \frac{B_1}{2}(p_2-q_2)+\frac{(B_2-B_1)}{4}(p_1^2-q_1^2). \quad (45)$$

It follows from (38), (39) and (45) that

$$a_3 = \frac{\tau^2 B_1^2(p_1^2+q_1^2)}{32(2-\lambda)\Gamma_2^2}+\frac{\tau B_1(p_2-q_2)}{12(3-\lambda)\Gamma_3}.$$

Applying Lemma (1) once again for the coefficients p_2 and q_2, we readily get

$$|a_3| \le \frac{|\tau|^2 B_1^2}{4(2-\lambda)^2\Gamma_2^2}+\frac{|\tau|B_1}{3(3-\lambda)\Gamma_3}.$$

This completes the proof of Theorem (2).

By putting $\lambda = 0$ in Theorem (2), we have the following corollary

Corollary 5. Let the function $f(z)$ given by (1) be in the class $\mathcal{G}_\Sigma^{s,b}(\tau;\phi)$. Then

$$|a_2| \le \frac{|\tau|B_1\sqrt{B_1}}{\sqrt{|16(B_1-B_2)\Gamma_2^2+9\tau B_1^2\Gamma_3|}} \text{and} |a_3| \le \frac{|\tau|^2 B_1^2}{16\Gamma_2^2}+\frac{|\tau|B_1}{9\Gamma_3}.$$

Taking $s = 0$, we have $\Gamma_n = 1 \quad (n \ge 2)$ in Theorem (2), and we can state the coefficient estimates for the functions in the subclass $\mathcal{N}_\Sigma(\tau,\lambda;\phi)$.

Corollary 6. Let the function $f(z)$ given by (1) be in the class $\mathcal{N}_\Sigma(\tau,\lambda;\phi)$. Then

$$|a_2| \le \frac{|\tau|B_1\sqrt{B_1}}{\sqrt{|4(2-\lambda)^2(B_1-B_2)+\tau(9-11\lambda+4\lambda^2)B_1^2|}} \text{ and} |a_3| \le \frac{|\tau|^2 B_1^2}{4(2-\lambda)^2}+\frac{|\tau|B_1}{3(3-\lambda)}.$$

Taking $\lambda = 0$ in Corollary (6), we get the following corollary

Corollary 7. Let the function $f(z)$ given by (1) be in the class $\mathcal{G}_\Sigma(\tau;\phi)$. Then

$$|a_2| \le \frac{|\tau|B_1\sqrt{B_1}}{\sqrt{|16(B_1-B_2)+9\tau B_1^2|}} \text{and} |a_3| \le \frac{|\tau|^2 B_1^2}{16}+\frac{|\tau|B_1}{9}.$$

Remark 4. Putting $\lambda = 1$ in Corollary (6), we obtain the results given by Corollary (4).

III. COROLLARIES AND ITS CONSEQUENCES

For the class of strongly starlike functions, the function ϕ is given by

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^\alpha = 1+2\alpha z+2\alpha^2 z^2+\cdots \quad (0<\alpha\le 1), \quad (46)$$

which gives $B_1 = 2\alpha$ and $B_2 = 2\alpha^2$.

Corollary 8. By choosing $\phi(z)$ of the form (46), we state the following results

(i) for function $f \in \mathcal{H}_\Sigma^{s,b}(\tau,\lambda;\phi)$, by Theorem (1),

$$|a_2| \le \frac{2|\tau|\alpha}{\sqrt{|(1+\lambda)^2(1-\alpha-2\tau\alpha)\Gamma_2^2+4\tau\alpha(1+2\lambda)\Gamma_3|}} \text{and} |a_3| \le \frac{4|\tau|^2\alpha^2}{(1+\lambda)^2\Gamma_2^2}+\frac{|\tau|\alpha}{(1+2\lambda)\Gamma_3}.$$

(ii) for function $f \in \mathcal{N}_\Sigma^{s,b}(\tau,\lambda;\phi)$, by Theorem (2),

$$|a_2| \le \frac{2|\tau|\alpha}{\sqrt{|4[2\tau\alpha(\lambda^2-2\lambda)+(1-\alpha)(2-\lambda)^2]\Gamma_2^2+6\tau\alpha(3-\lambda)\Gamma_3|}} \quad \text{and} |a_3| \le \frac{|\tau|^2\alpha^2}{(2-\lambda)^2\Gamma_2^2}+\frac{2|\tau|\alpha}{3(3-\lambda)\Gamma_3}.$$

On the other hand if we take

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$$\phi(z) = \frac{1+(1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \cdots \quad (0 \le \beta < 1), \tag{47}$$

then we have $B_1 = B_2 = 2(1-\beta)$.

Corollary 9. By choosing $\phi(z)$ of the form (47), we state the following results

(i) for function $f \in \mathcal{H}_{\Sigma}^{s,b}(\tau,\lambda;\phi)$, by Theorem (1),

$$|a_2| \le \frac{|\tau|\sqrt{2(1-\beta)}}{\sqrt{|2\tau(1+2\lambda)\Gamma_3 - \tau(1+\lambda)^2\Gamma_2^2|}} \text{and } |a_3| \le \frac{4|\tau|^2(1-\beta)^2}{(1+\lambda)^2\Gamma_2^2} + \frac{|\tau|(1-\beta)}{(1+2\lambda)\Gamma_3}.$$

(ii) for function $f \in \mathcal{N}_{\Sigma}^{s,b}(\tau,\lambda;\phi)$, by Theorem (2),

$$|a_2| \le \frac{|\tau|\sqrt{2(1-\beta)}}{\sqrt{|3\tau(3-\lambda)\Gamma_3 + 4\tau(\lambda^2-2\lambda)\Gamma_2^2|}} \quad \text{and} \quad |a_3| \le \frac{|\tau|^2(1-\beta)^2}{(2-\lambda)^2\Gamma_2^2} + \frac{2|\tau|(1-\beta)}{3(3-\lambda)\Gamma_3}.$$

Corollary 10. Let $f(z)$ given by (1) be in the class $\mathcal{S}_{\Sigma}^{s,b}(\tau;\phi)$ and $\phi(z)$ is of the form (46), then from Theorem (1), we have

$$|a_2| \le \frac{2|\tau|\alpha}{\sqrt{|(1-\alpha-2\tau\alpha)\Gamma_2^2 + 4\tau\alpha\Gamma_3|}} \text{and} |a_3| \le \frac{4|\tau|^2\alpha^2}{\Gamma_2^2} + \frac{|\tau|\alpha}{\Gamma_3}.$$

Corollary 11. Let $f(z)$ given by (1) be in the class $\mathcal{S}_{\Sigma}^{s,b}(\tau;\phi)$ and $\phi(z)$ is of the form (47), then from Theorem (1), we have

$$|a_2| \le \frac{|\tau|\sqrt{2(1-\beta)}}{\sqrt{|2\tau\Gamma_3 - \tau\Gamma_2^2|}} \text{and} |a_3| \le \frac{4|\tau|^2(1-\beta)^2}{\Gamma_2^2} + \frac{|\tau|(1-\beta)}{\Gamma_3}.$$

Remark 5. Putting $s = 0$ and $\tau = 1$ in Corollary (10) and Corollary (11), we obtain the corresponding results given by Li and Wang [9].

Corollary 12. Let $f(z)$ given by (1) be in the class $\mathcal{K}_{\Sigma}^{s,b}(\tau;\phi)$ and $\phi(z)$ is of the form (46), then from Theorem (1), we have

$$|a_2| \le \frac{|\tau|\alpha}{\sqrt{|(1-\alpha-2\tau\alpha)\Gamma_2^2 + 3\tau\alpha\Gamma_3|}} \text{and} |a_3| \le \frac{|\tau|^2\alpha^2}{\Gamma_2^2} + \frac{|\tau|\alpha}{3\Gamma_3}.$$

Corollary 13. Let $f(z)$ given by (1) be in the class $\mathcal{K}_{\Sigma}^{s,b}(\tau;\phi)$ and $\phi(z)$ is of the form (47), then from Theorem (1), we have

$$|a_2| \le \frac{|\tau|\sqrt{2(1-\beta)}}{\sqrt{|6\tau\Gamma_3 - 4\tau\Gamma_2^2|}} \text{and} |a_3| \le \frac{|\tau|^2(1-\beta)^2}{\Gamma_2^2} + \frac{|\tau|(1-\beta)}{3\Gamma_3}.$$

Remark 6. Putting $s = 0$ and $\tau = 1$ in Corollary (12) and Corollary (13), we obtain the corresponding results given by Murugusundaramoorthy et al. [13].

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