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Differential Subordination and Superordination for Multivalent Functions Involving a Generalized Differential Operator

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ABSTRACT: In this paper, we deduce some subordination and superordination outcomes involving the generalized differential operator $D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)$ for certain multivalent analytic functions in the open unit disk. These outcomes are applied to obtain differential sandwich theorems.

KEYWORDS: Analytic function; multivalent function; differential subordination; differential superordination; sandwich theorem; generalized differential operator.

I. INTRODUCTION

Let$\mathcal{H}=\mathcal{H}(U)$ symbolize the class of analytic functions in the open unit disk $U=\{z\in\mathbb{C}: |z|<1\}$ and let $\mathcal{H}[a,p]$ symbolize the subclass of the function $f\in\mathcal{H}$ of the shape:

$$f(z)=a+a_pz^p+a_{p+1}z^{p+1}+\cdots \quad (a\in\mathbb{C}; p\in\mathbb{N}=\{1,2,\dots\}). \tag{1}$$

Also, let $\mathcal{A}(p)$ be the subclass of $\mathcal{H}$ consisting of functions of the shape:

$$f(z)=z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}(p\in\mathbb{N}=\{1,2,\dots\}). \tag{2}$$

Let $f,g\in\mathcal{H}$, if there exists a Schwarz function w analytic in U with $w(0)=0$ and $|w(z)|<1$ $(z\in U)$ such that $f(z)=g\big(w(z)\big)$, then the function f is invited subordinate to g, or g is invited superordinate to f, In such a case we write $f\prec g$ or $f(z)\prec g(z)$ $(z\in U)$. If g is univalent in U, then $f\prec g$ if and only if $f(0)=g(0)$ and $f(U)\subset g(U)$.

Let $p,h\in\mathcal{H}$ and $\varphi(r,s,t;z):\mathbb{C}^3\times U\to\mathbb{C}$. If p and $\varphi(p(z),zp'(z),z^2p''(z);z)$ are univalent functions in U and if p satisfies the second-order superordination

$$h(z)\prec\varphi(p(z),zp'(z),z^2p''(z);z), \tag{3}$$

then p is invited a solution of the differential superordination (3). (If f is subordinate to g, then g is superordinate to f). An analytic function q is invited a subordinant of (3), if $q\prec p$ for all the function p satisfying (3). An univalent subordinant $\tilde{q}$ that satisfies $q\prec\tilde{q}$ for all the subordinants q of (3) is invited the best subordinant. Recently, Miller and Mocanu [1] gained conditions on the functions h,q and φ for which the following modulation holds:

$$h(z)\prec\varphi(p(z),zp'(z),z^2p''(z);z)\Rightarrow q(z)\prec p(z).$$

Now, (x_n) denotes the Pochhammer symbol defined by

$$(x_n)=\frac{\Gamma(x+n)}{\Gamma(x)}=\begin{cases}1, & n=0,\\ x(x-1)\dots(x+n-1), & n=\{1,2,3,\dots\}.\end{cases}$$

El-Yagubi and Darus [2] defined a generalized differential operator, as follows:

$$D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1):\mathcal{A}(p)\to\mathcal{A}(p)$$

$$D^{m,b}_{\lambda_1,\lambda_2,p}f(z)=z^p+\sum_{n=1}^{\infty}\left[\frac{p+(\lambda_1+\lambda_2)n+b}{p+\lambda_2n+b}\right]^m\frac{(a_1)_n\dots(a_r)_n}{(b_1)_n\dots(b_s)_n}\frac{a_{p+n}z^{p+n}}{n!}, \tag{4}$$

where $m,b,r,s\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\lambda_2\geq\lambda_1\geq0$ and $a_i\in\mathbb{C}, b_q\in\mathbb{C}\setminus\{0,-1,-2,\dots\}, (i=1,\dots,r,\ q=1,\dots,s), r\leq s+1$.

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It follows from (4) that

$$\lambda_1 z\left(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)\right)' = (p+\lambda_2 n+b)\,D^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) - (p+\lambda_2 n-p\lambda_1+b)\,D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z). \quad (5)$$

It should be noted that the linear operator $D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)$ is a generalization of many other linear operators considered earlier. In particular:

(1) For $\lambda_2 = b = 0$, the operator $D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)$ reduces to the operator was given by Selvaraj and Karthikeyan [3].
(2) For $m = 0$, the operator $D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)$ reduces to the operator was given by El-Ashwah [4].
(3) For $m = 0$, and $p = 1$, the operator $D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)$ reduces to the well-known operator introduced by Dziok and Srivastava [5].
(4) For $\mathrm{m} = 0, \mathrm{r} = 2, \mathrm{s} = 1$ and $\mathrm{p} = 1$, we gain the operator which was given by Hohlov [6].
(5) For $r = 1, s = 0, a_1 = 1, \lambda_1 = 1, \lambda_2 = b = 0$ and $p = 1$, we get the Salagean derivative operator [7].

The main object of the present paper is to find sufficient conditions for certain normalized analytic functions f to satisfy

$$q_1(z) \prec \left(\frac{z^p}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}\right)^{\mu} \prec q_2(z)$$

and

$$q_1(z) \prec \left(\frac{tD^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) + (1-t)D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}{z^p}\right)^{\mu} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$.

II. PRELIMINARIES

In order to manifest our leading results, we require the following definition and lemmas.

Definition (1) [8]: Denote by Q the set of all functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$\mathrm{E}(f) = \left\{\zeta \in \partial U : \lim_{z\to\zeta} f(z) = \infty\right\} \quad (6)$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$.

Lemma (1)[1]: Let q be a convex univalent function in U and let $\alpha \in \mathbb{C}, \beta \in \mathbb{C}\backslash \{0\}$ with

$$Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -Re\left(\frac{\alpha}{\beta}\right)\right\}.$$

If p is analytic in U and

$$\alpha p(z) + \beta z p'(z) \prec \alpha q(z) + \beta z q'(z), \quad (7)$$

then $p \prec q$ and q is the best dominant of (7).

Lemma(2) [9]: Let q be univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(u)$. Set $Q(z) = zq'(z)\,\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that

(1) $Q(z)$ is starlike univalent in U,
(2) $Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0\ for\ z \in U$.

If p is analytic in U, with $p(0) = q(0), p(U) \subset D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \quad (8)$$

then $p \prec q$ and q is the best dominant of (8).

Lemma (3) [1]: Let q be convex univalent in U and let $\beta \in \mathbb{C}$. Further assume that $Re(\beta) > 0$. If $p \in \mathcal{H}[q(0), 1] \cap Q$ and $p(z) + \beta z p'(z)$ is univalent in U, then

$$q(z) + \beta z q'(z) \prec p(z) + \beta z p'(z), \quad (9)$$

which implies that $q \prec p$ and q is the best subordinant of (9).

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Lemma(4) [9]: Let q be convex univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$. Suppose that

(1) $Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} > 0 \; for \; z \in U,$

(2) $Q(z) = zq'(z)\phi(q(z))$ is starlike univalent in U.

If $p \in \mathcal{H}[q(0),1] \cap Q$, with $p(U) \subset D, \theta(p(z)) + zp'(z)\phi(p(z))$ is univalent in U and

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \phi(p(z)) + zp'(z)\phi(p(z)), \tag{10}$$

then $q \prec p$ and q is the best subordinant of (10).

III. SUBORDINATION RESULTS

Theorem (1): Let $q(z)$ be convex univalent in U with $q(0) = 1, \eta \in \mathbb{C}/\{0\},\ \mu > 0$ and suppose that

$$Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -Re\left(\frac{\mu(p + \lambda_2 n + b)}{\eta\lambda_1}\right)\right\}. \tag{11}$$

If

$$\nabla_1(z) = (1+\eta)\left(\frac{z^p}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}\right)^{\mu} - \eta\left(\frac{z^p}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}\right)^{\mu}\left(\frac{D^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}\right) \tag{12}$$

and

$$\nabla_1(z) \prec q(z) + \frac{\eta\lambda_1}{\mu(p + \lambda_2 n + b)}zq'(z), \tag{13}$$

then

$$\left(\frac{z^p}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}\right)^{\mu} \prec q(z) \tag{14}$$

and $q(z)$ is the best dominant of (13).

Proof: Define the analytic function $p(z)$ by

$$p(z) = \left(\frac{z^p}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}\right)^{\mu} \tag{15}$$

Differentiating (15) logarithmically with respect to z, we have

$$\frac{zp'(z)}{p(z)} = \mu\left[p - \frac{z\left(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)\right)'}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}\right]. \tag{16}$$

Now, using the identity (5), we obtain the following

$$\frac{zp'(z)}{p(z)} = \frac{\mu(p + \lambda_2 n + b)}{\lambda_1}\left(1 - \frac{D^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}\right).$$

Therefore,

$$\frac{\lambda_1}{\mu(p + \lambda_2 n + b)}zp'(z) = \left(\frac{z^p}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}\right)^{\mu}\left(1 - \frac{D^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}\right).$$

Thus, the subordination (13) is equivalent to

$$p(z) + \frac{\eta\lambda_1}{\mu(p + \lambda_2 n + b)}zp'(z) \prec q(z) + \frac{\eta\lambda_1}{\mu(p + \lambda_2 n + b)}zq'(z).$$

Applying Lemma (1) with $\beta = \frac{\eta\lambda_1}{\mu(p+\lambda_2 n+b)}$ and $\alpha = 1$, we obtain (14).

Putting $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ in Theorem (1), we get the following result.

Corollary (1): Let $\eta \in \mathbb{C}/\{0\}$ and $-1 \le B < A \le 1$. Also, suppose that

$$Re\left(\frac{1 - Bz}{1 + Bz}\right) > \max\left\{0, -Re\left(\frac{\mu(p + \lambda_2 n + b)}{\eta\lambda_1}\right)\right\}.$$

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If $f \in \mathcal{A}(p)$ satisfies the following subordination condition:

$$\nabla_1(z) \prec \frac{1+Az}{1+Bz} + \frac{\eta\lambda_1}{\mu(p+\lambda_2 n+b)}\frac{(A-B)z}{(1+Bz)^2},$$

where $\nabla_1(z)$ given by (12), then

$$\left(\frac{z^p}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}\right)^{\mu} \prec \frac{1+Az}{1+Bz}$$

and the function $\frac{1+Az}{1+Bz}$ is the best dominant.
Taking $A = 1$ and $B = -1$ in Corollary (1), we get the following result.

Corollary (2): Let $\eta \in \mathbb{C}/\{0\}$ and Suppose that

$$Re\left(\frac{1+z}{1-z}\right) > \max\left\{0, -Re\left(\frac{\mu(p+\lambda_2 n+b)}{\eta\lambda_1}\right)\right\}.$$

If $f \in \mathcal{A}(p)$ satisfies the following subordination:

$$\nabla_1(z) \prec \frac{1+z}{1-z} + \frac{\eta\lambda_1}{\mu(p+\lambda_2 n+b)}\frac{2z}{(1-z)^2},$$

where $\nabla_1(z)$ given by (12), then

$$\left(\frac{z^p}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}\right)^{\mu} \prec \frac{1+z}{1-z}$$

and the function $\frac{1+z}{1-z}$ is the best dominant.

Theorem (2): Let $q(z)$ be univalent in U with $q(0) = 1,\ q(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ is starlike in U, let $\mu, \eta \in \mathbb{C}/\{0\}$and $u, v, \xi \in \mathbb{C}$. Let $f \in \mathcal{A}(p)$ and suppose that f and g satisfy the next two conditions:

$$\frac{tD^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) + (1-t)D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}{z^p} \neq 0 \quad (z \in U,\ 0 \le t \le 1) \tag{17}$$

and

$$Re\left\{1 + \frac{v}{\eta}q(z) + \frac{2\xi}{\eta}[q(z)]^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0 \tag{18}$$

If

$$\begin{aligned}\nabla_2(z) = u + v\left(\frac{tD^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) + (1-t)D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}{z^p}\right)^{\mu} \\ +\xi\left(\frac{tD^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) + (1-t)D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}{z^p}\right)^{2\mu} \\ +\eta\mu\left[\frac{tz\left(D^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)\right)' + (1-t)z\left(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)\right)'}{tD^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) + (1-t)D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)} - p\right].\end{aligned} \tag{19}$$

and

$$\nabla_2(z) \prec u + vq(z) + \xi[q(z)]^2 + \eta\frac{zq'(z)}{q(z)}, \tag{20}$$

then

$$\left(\frac{tD^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) + (1-t)D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}{z^p}\right)^{\mu} \prec q(z), \tag{21}$$

and q is the best dominant of (20).
Proof: Define the analytic function p by

$$p(z) = \left(\frac{tD^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) + (1-t)D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}{z^p}\right)^{\mu}. \tag{22}$$

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Then p is analytic in U and $p(0) = 1$, differentiating (22) logarithmically with respect to z, we get

$$\frac{zp'(z)}{p(z)} = \mu\left[\frac{tz\left(D^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)\right)' + (1-t)z\left(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)\right)'}{tD^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) + (1-t)D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)} - p\right]. \quad (23)$$

By setting

$$\theta(w) = u + vw + \xi w^2 \text{ and } \phi(w) = \frac{\eta}{w}, \quad (w \in \mathbb{C}/\{0\}),$$

we see that $\theta(w)$is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C}/\{0\}$ and that $\phi(w) \neq 0, w \in \mathbb{C}/\{0\}$. Also, we get

$$Q(z) = zq'(z)\,\phi(q(z)) = \eta\frac{zq'(z)}{q(z)}, \quad (z \in U),$$

and

$$h(z) = \theta(q(z)) + Q(z) = u + vq(z) + \xi[q(z)]^2 + \eta\frac{zq'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike in U and, that

$$Re\frac{zh'(z)}{Q(z)} = Re\left\{1 + \frac{v}{\eta}q(z) + \frac{2\xi}{\eta}[q(z)]^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0 \qquad (z \in U).$$

By making use of (23), the hypothesis (20) can be equivalently written as

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \phi(q(z)) + zq'(z)\phi(q(z)),$$

thus, by applying Lemma (2), the proof is completed.

Theorem (3): Let $q(z)$ be univalent in U with $q(0) = 1$, let $\mu, \eta \in \mathbb{C}/\{0\}$and $v, \xi \in \mathbb{C}$. Let $f(z) \in \mathcal{A}(p)$ and suppose that f and g satisfy the next two conditions:

$$\frac{tD^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) + (1-t)D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}{z^p} \neq 0 \quad (z \in U,\ 0 \le t \le 1) \quad (24)$$

And

$$Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -Re\left(\frac{v}{\eta}\right)\right\} (z \in U). \quad (25)$$

If

$$\nabla_3(z) = \left(\frac{tD^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) + (1-t)D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}{z^p}\right)^{\mu} \times$$

$$\left[v + \eta\mu\left(\frac{tz\left(D^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)\right)' + (1-t)\left(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)\right)'}{tD^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) + (1-t)D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)} - p\right)\right] + \xi \quad (26)$$

and

$$\nabla_3(z) \prec vq(z) + \eta zq'(z) + \xi, \quad (27)$$

then

$$\left(\frac{tD^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) + (1-t)D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}{z^p}\right)^{\mu} \prec q(z), \quad (28)$$

and q is the best dominant of (27).

Proof: Let the function p be defined on U by (16). Then a computation shows that

$$zp'(z) = \mu\left(\frac{tD^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) + (1-t)D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}{z^p}\right)^{\mu} \times$$

$$\left[\frac{tz\left(D^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)\right)' + (1-t)z\left(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)\right)'}{tD^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) + (1-t)D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)} - p\right]. \quad (29)$$

By setting

$$\theta(w) = vw + \xi, \quad \phi(w) = \eta, \quad (w \in \mathbb{C}),$$

we see that $\theta(w)$, $\phi(w)$ are analytic in $\mathbb{C}$ and that $\phi(w) \neq 0$. Also, we get

$$Q(z) = zq'(z)\,\phi(q(z)) = \eta zq'(z), \quad (z \in U),$$

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and

$$h(z) = \theta(q(z)) + Q(z) = vq(z) + \eta z q'(z) + \xi \qquad (z \in U).$$

From the assumption (25) we see that $Q(z)$ is starlike in U and, that

$$Re\frac{zh'(z)}{Q(z)} = Re\left\{\frac{v}{\eta} + \frac{zq''(z)}{q'(z)} + 1\right\} > 0 \qquad (z \in U),$$

and then, by using Lemma (2) we deduce that the subordination (27) implies $p(z) \prec q(z)$, and the function q is the best dominant of (27).

IV. SUPERORDINATION RESULTS

Theorem (4): Let q be convex in U with $q(0) = 1, \mu > 0$ and $Re\{\eta\} > 0$. Let $f \in \mathcal{A}(p)$ satisfies

$$\left(\frac{z^p}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}\right)^{\mu} \in \mathcal{H}[q(0),1] \cap Q.$$

If the function $\nabla_1(z)$ given by (12) is univalent in U, and

$$q(z) + \frac{\eta\lambda_1}{\mu(p+\lambda_2 n + b)} zq'(z) \prec \nabla_1(z), \tag{30}$$

then

$$q(z) \prec \left(\frac{z^p}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}\right)^{\mu} \tag{31}$$

and q is the best subordinant of (30).

Proof: Define the analytic function $p(z)$ by

$$p(z) = \left(\frac{z^p}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}\right)^{\mu}. \tag{32}$$

Differentiating (32) logarithmically with respect to z, we have

$$\frac{zp'(z)}{p(z)} = \mu\left[p - \frac{z\left(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)\right)'}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}\right]. \tag{33}$$

After some computations and using the identity (5), from (33), we have

$$\nabla_1(z) = p(z) + \frac{\eta\lambda_1}{\mu(p+\lambda_2 n + b)} zp'(z),$$

and now, by using Lemma (3), we get the desired result.

Putting $q(z) = \frac{1+Az}{1+Bz} (-1 \le B < A \le 1)$ in Theorem (4), we get the following corollary.

Corollary (3): Let $-1 \le B < A \le 1,\ \mu > 0$ and $Re\{\eta\} > 0$. Also let

$$\left(\frac{z^p}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}\right)^{\mu} \in \mathcal{H}[q(0),1] \cap Q.$$

If the function $\nabla_1(z)$ given by (12) is univalent in U, and $f \in \mathcal{A}(p)$ satisfies the following superordination condition:

$$\frac{1+Az}{1+Bz} + \frac{\eta\lambda_1}{\mu(p+\lambda_2 n + b)}\frac{(A-B)z}{(1+Bz)^2} \prec \nabla_1(z),$$

then

$$\frac{1+Az}{1+Bz} \prec \left(\frac{z^p}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}\right)^{\mu}$$

and the function $\frac{1+Az}{1+Bz}$ is the best subordinant.

Theorem (5): Let q be convex univalent in U with $q(0) = 1,\ q(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ is starlike in U, let $\mu, \eta \in \mathbb{C}/\{0\}$ and $u, v, \xi \in \mathbb{C}$. Further assume that q satisfies

$$Re\left\{(v + 2\xi q(z))\frac{q(z)q'(z)}{\eta}\right\} > 0 \qquad (z \in U). \tag{34}$$

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Let $f(z) \in \mathcal{A}(p)$ and suppose that $f(z)$ satisfies the next conditions:

$$\frac{tD^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) + (1-t)D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}{z^p} \neq 0 \quad (z \in U,\ \ 0 \leq t \leq 1) \tag{35}$$

and

$$\left(\frac{tD^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) + (1-t)D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}{z^p}\right)^{\mu} \in \mathcal{H}[q(0),1] \cap Q. \tag{36}$$

If the function $\nabla_2(z)$ given by (19) is univalent in U, and

$$u + vq(z) + \xi[q(z)]^2 + \eta\frac{zq'(z)}{q(z)} \prec \nabla_2(z), \tag{37}$$

then

$$q(z) \prec \left(\frac{tD^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) + (1-t)D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}{z^p}\right)^{\mu}, \tag{38}$$

and q is the best subordinant of (37).

Proof: Let the function $p(z)$ be defined on U by (22). Then a computation shows that

$$\frac{zp'(z)}{p(z)} = \mu\left[\frac{tz\left(D^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)\right)' + (1-t)z\left(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)\right)'}{tD^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) + (1-t)D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)} - p\right]. \tag{39}$$

By setting

$$\theta(w) = u + vw + \xi w^2 \ \ and \ \ \ \phi(w) = \frac{\eta}{w}, \quad (w \in \mathbb{C}/\{0\}),$$

we see that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$is analytic in $\mathbb{C}/\{0\}$ and that $\phi(w) \neq 0,\ w \in \mathbb{C}/\{0\}$. Also, we get

$$Q(z) = zq'(z)\,\phi(q(z)) = \eta\frac{zq'(z)}{q(z)}, \quad (z \in U).$$

It is observe that $Q(z)$ is starlike in U and, that

$$Re\frac{\theta'(q(z))}{\phi(q(z))} = Re\left\{(v + 2\xi q(z))\frac{q(z)q'(z)}{\eta}\right\} > 0 \qquad (z \in U).$$

By making use of (39) the hypothesis (37) can be equivalently written as

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(p(z)) + zp'(z)\phi(p(z)),$$

thus, by applying Lemma (4), the proof is completed.

Using arguments similar to those of the proof of Theorem (3), and then by applying Lemma (4), we obtain the following result.

Theorem (6): Let q be convex in U with $q(0) = 1$, let $\mu, \eta \in \mathbb{C}/\{0\}$ and $v, \xi \in \mathbb{C}$ and $Re\left\{\frac{v}{\eta}q'(z)\right\} > 0$. Let $f \in \mathcal{A}(p)$ and suppose that $f(z)$ satisfies the next conditions:

$$\frac{tD^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) + (1-t)D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}{z^p} \neq 0 \quad (z \in U,\ \ 0 \leq t \leq 1) \tag{40}$$

and

$$\left(\frac{tD^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) + (1-t)D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}{z^p}\right)^{\mu} \in \mathcal{H}[q(0),1] \cap Q. \tag{41}$$

If the function $\nabla_3(z)$ given by (26) is univalent in U, and

$$vq(z) + \eta zq'(z) + \xi \prec \nabla_3(z), \tag{42}$$

then

$$q(z) \prec \left(\frac{tD^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) + (1-t)D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}{z^p}\right)^{\mu} \tag{43}$$

and q is the best subordinant of (42).

V. SANDWICH RESULTS

By combining Theorem (1) with Theorem (4), we obtain the following sandwich theorem:

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Theorem (7): Let q_1 and q_2 be two convex functions in U, $q_1(0) = q_2(0) = 1$ and q_2 satisfies (11), $\mu > 0, \eta \in \mathbb{C}$ with $Re\{\eta\} > 0$. If $f \in \mathcal{A}(p)$ such that $\left(\frac{z^p}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}\right)^\mu \in \mathcal{H}[1,1] \cap Q, \nabla_1(z)$ is univalent in U and satisfies

$$q_1(z) + \frac{\eta\lambda_1}{\mu(p+\lambda_2 n+b)} zq_1'(z) \prec \nabla_1(z) \prec q_2(z) + \frac{\eta\lambda_1}{\mu(p+\lambda_2 n+b)} zq_2'(z), \quad (44)$$

where $\nabla_1(z)$ is given by (12), then

$$q_1(z) \prec \left(\frac{z^p}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}\right)^\mu \prec q_2(z),$$

where q_1 and q_2 are , respectively, the best subordinant and the best dominant of (44).
By combining Theorem (2) with Theorem (5), we obtain the following sandwich theorem:

Theorem (8): Let q_i be two convex functions in U, such that $q_i(0) = 1$, $q_i(z) \neq 0$ and $\frac{zq_i'(z)}{q_i(z)} (i = 1,2)$ is starlike in U, let $\mu, \eta \in \mathbb{C}/\{0\}$ and $u, v, \xi \in \mathbb{C}$. Further assume that q_1 satisfies (34), and $q_2$satisfies (18). Let $f \in \mathcal{A}(p)$ and suppose that f satisfies the next conditions:

$$\frac{tD^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) + (1-t)D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}{z^p} \neq 0 \quad (z \in U,\ 0 \leq t \leq 1),$$

and

$$\left(\frac{tD^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) + (1-t)D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}{z^p}\right)^\mu \in \mathcal{H}[1,1] \cap Q.$$

If the function $\nabla_2(z)$ given by (19) is univalent in U, and

$$u + vq_1(z) + \xi[q_1(z)]^2 + \eta\frac{zq_1'(z)}{q_1(z)} \prec \nabla_2(z) \prec u + vq_2(z) + \xi[q_2(z)]^2 + \eta\frac{zq_2'(z)}{q_2(z)}, \quad (45)$$

then

$$q_1(z) \prec \left(\frac{tD^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) + (1-t)D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}{z^p}\right)^\mu \prec q_2(z),$$

where q_1 and q_2 are , respectively, the best subordinant and the best dominant of (45).
By combining Theorem (3) with Theorem (6), we obtain the following sandwich theorem:
Theorem (9): Let q_1 and q_2 be two convex functions in U, with $q_1(0) = q_2(0) = 1$, let $\mu, \eta \in \mathbb{C}/\{0\}$ and $v, \xi \in \mathbb{C}$ with $Re\left\{\frac{v}{\eta}q_1'(z)\right\} > 0$ and q_2 satisfies (25). Let $f \in \mathcal{A}(p)$ and suppose that f satisfies the next conditions:

$$\frac{tD^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) + (1-t)D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}{z^p} \neq 0 \quad (z \in U,\ 0 \leq t \leq 1),$$

and

$$\left(\frac{tD^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) + (1-t)D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}{z^p}\right)^\mu \in \mathcal{H}[1,1] \cap Q.$$

If the function $\nabla_3(z)$ given by (26) is univalent in U, and

$$vq_1(z) + \eta zq_1'(z) + \xi \prec \nabla_3(z) \prec vq_2(z) + \eta zq_2'(z) + \xi, \quad (46)$$

then

$$q_1(z) \prec \left(\frac{tD^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) + (1-t)D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)}{z^p}\right)^\mu \prec q_2(z),$$

where q_1 and q_2 are , respectively, the best subordinant and the best dominant of (46).

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