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Deegan Packel Index for fuzzy simple games in Multilinear Extension Form

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ABSTRACT:In this paper, we introduce the fuzzy versions of Deegan Packel Index (DPI)and obtain the respective characterizations for fuzzy simple games. The class of games in multilinear extension form is proposed and the corresponding power indices is formulated.

KEYWORDS: Fuzzy simple games, Deegan packel power index, fuzzy sets, fuzzy coalition

I.INTRODUCTION

Power indices study the various socio-political and economic issues related to a simple game. In a simple game the value of a winning coalition is taken as 1 and all other coalitions (loosing coalitions) generate zero value. The Deegan-Packel Index (DPI in short) is important power indices introduced by Deegan and Packel [3]. Note that the DPI measure in some sense, the influence of a player in her decision to shift her loyalty from the minimal winning coalitions. A minimal winning coalition is one from where removal of any player transforms it into a loosing coalition. Here each member of a minimal winning coalition is named as a voter. The DPI of voter $i$is equal to the sum of the inverse of the cardinality of each coalition S in the sets of minimal winning coalitions where she belongs, divided by the cardinality of the sets of minimal winning coalitions. In this paper we propose the Deegan-Packel index (DPI) for a simple game with fuzzy coalitions (a fuzzy simple game in short) and obtain the respective characterizations. We call the fuzzy DPI. In [1] multilinear extension of a simple game is defined. We introduce a new class of fuzzy simple games in multilinear extension form in the line of [1] and obtain the fuzzy DPI for this class of games. In the rest of the paper we follow the characterizations of the PGI and DPI due to [1, 7].

The rest of the paper is organized as follows. Section 2 presents the preliminary concepts about the DPI in crisp settings. In Section 3, the notions of fuzzy DPI is introduced for fuzzy simple games. Section 4 introduces the new class of fuzzy simple games in multilinear extension form. The fuzzy DPI for this class of simple games is proposed followed by an illustrative example in Section 5. Finally, Section 6 brings out some concluding remarks.

II.PRELIMINARIES

In this section we mention the necessary preliminaries concepts relating to cooperative games in crisp and their power indices especially DPI in crisp settings which bear similar notions from [7].

A.DEEGAN-PACKEL INDEX

In the following we define the Deegan-Packel index for the class G^Nas follows.
Definition 2.1. A function $f : G^N \to \mathbb{R}^n$defines a value due to [7] if for every $S \in 2^N$it satisfies the following axioms.
Axiom d1 (Efficiency) : If $(N; v) \in G^N$, it holds that,

$$\sum_{i \in N} f_i\,(N,v) \;=\; 1$$

Axiom d2 (Null Player) : If player $i \;\in N$is null for $(N,v)\; \in G^N$, then,

$$f_i(N,v)(S) \;=\; 0$$

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Axiom d3 (Symmetry): If $i, j \in N$are two symmetric players in a game $(N, v) \in G^N$ then,

$$f_i(N,v)(S) = f_j(N,v)(S)$$

Axiom d4 (DP-minimal monotonicity): If $(N,v),(N,w) \in G^N$, it holds that for all player $i \in N$such that $M_i(v) \subseteq M_i(w)$,

$$f_i(N,w)|M(w)| \geq f_i(N,v)|M(v)|$$

Theorem 2.11. There exist a unique Deegan-Packel index f on GN that satisfies Axiom d1-Axiom d4 and is given by,

$$f_i(N,v) = \frac{1}{M(v)} \sum_{S \in M_i(v)} \frac{1}{|S|}$$

It can be rewritten as follows.
Theorem 2.12There exist a unique Deegan-Packel index $f$on $G^N$for $S \in 2^N$that satisfies Axiom d1-Axiom d4 and is given by,

$$f_i(N,v)(S) = \frac{1}{M^S(v)} \sum_{T \in M_i^S(v)} \frac{1}{|T|}$$

Where $M^S(v) and\ M_i^S(v)$denotes the sets of minimal winning coalitions in S andthe sets of minimal winning coalitions containing the player i in Srespectively. In theremaining part of the paper we use Theorem 2.11instead of Theorem 2.12.

III. FUZZY DPI FOR FUZZY SIMPLE GAMES

Definition 3.1. A TU game w with fuzzy coalition [8] is said to be a fuzzy simple game,if for each fuzzy coalition $S \in L(N)$, we have $w(S) \in [0,1]$and $w$satisfy the following twoconditions.
1. $w(S_\emptyset) = 0, w(S_N) = 1$
2. Monotonicity: $w(S) \leq w(T)\ if\ S \subseteq T$
Let $G_{FC}(N)$ denote the class of all fuzzy simple games. Prior to the definition of public good index in fuzzy settings we define the following.

Definition 3.2. Given $\alpha \in (0,1]$. If $S \in L(N)$and $w \in G_{FC}(N)$, the player $i \in N$is said to be αnull for $w$in Sif $w(T \cup I) = w(T)$for all $T \in L(S)$with $T(j) > \alpha$when $j \neq i$, $T(i) < \alpha$ and all $I \in L(N)$such that $I(i) > \alpha\ and\ I(j) < \alpha$when $j \neq i$.

Definition 3.3. If $S \in L(N)$and $w \in G_{FC}(N)$, the players $i, j \in N$is said to be symmetric for $w$in Sif $w(T \cup I) = w(T \cup J)$for all $T \in L(S)$with $T(i) = 0 = T(j)$and all $I, J \in L(N)$such that $I(i) = J(j)$and $I(j) = 0$when $j \neq i$and $J(i) = 0$when $i \neq j$.
In the following we define the fuzzy winning coalition and minimal fuzzy winning coalition as follows.

Definition 3.4. Given $\alpha \in (0,1]$, let $S \in L(N)$and $w \in G_{FC}(N)$, a fuzzy coalition $T \in L(S)$is said to be α winning coalition if $w(T) \geq \alpha$ In analogy to this a fuzzy coalition $T \epsilon L(S)$is said to be loosing if $w(T) < \alpha$.

Definition 3.5. Let $S \epsilon L(N)$and $w \in G_{FC}(N)$, an α winning coalition $T \in L(S)$is said to beα minimal winning coalition if every proper subset of $T \in L(S)$is a loosing coalition.
We denote the set of αwinning coalitions for $w$in $S \in L(N)$ by $W(S_w^\alpha)$ and the setof α minimal winning coalitions $by\ M(S_w^\alpha)$. In the rest of the paper we denote the set ofminimal winning coalitions containing i by $M(S_{w_i}^\alpha)$.
A.FUZZY DPI

Definition 3.6. A function $f : G_{FC}(N) \to (\mathbb{R}^n)^{L(N)}$is said to be a Deegan-Packel indexin fuzzy settings in $G_{FC}(N)$if it satisfies the follwing axioms.
Axiom F1 (Efficiency): If $w \in G_{FC}(N)$ and $A \in L(N)$, then

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$$\sum_{i\in N} f_i\, w(A) = w(A)$$

Axiom F2 (Null player Axiom): If player $i \in SuppA$is a null player for $w \in G_{FC}(N)$ in$A \in L(N)$ then

$$f_i(w)(A) = 0$$

Axiom F3 (Symmetry): For every pair of fuzzy symmetric player $i, j \in SuppA$and forevery $w \in G_{FC}(N)$ we must have

$$f_i(w)(A) = f_j(w)(A)$$

Axiom F4 (DP-minimal monotonicity) : If $w, w' \in G_{FC}(N)$, it holds that

$$f_i(w')(A)\left|M(A^{\alpha}_{w'})\right| \geq f_i(w)(A)|M(A^{\alpha}_{w})|$$

for all $\alpha \in (0,1]$ such that $M(A^{\alpha}_{w_i}) \sqsubseteq M(A^{\alpha}_{w_i'})$

IV. FUZZY SIMPLE GAMES IN MULTILINEAR EXTENSION FORMFUZZY DPI FOR THIS CLASS

Define the fuzzy simple game $w_M: L(N) \rightarrow [0, 1]$ in multilinear extension form as follows

$$w_M(A) = \sum_{T\in M(A^{\alpha}_{w_M})} \prod_{t\in T} \mu_A(T)v(T) \quad (4.1)$$

Denote by $G^M_{FC}(N)$ the class of fuzzy simple games in multilinear extension form. For every$w_M \in G^M_{FC}(N)$we defne the function $f : G^M_{FC}(N) \rightarrow (\mathbb{R}^n)^{L(N)}$by

$$f_i^{\alpha}(w_M)(A) = \sum_{T\in M(A^{\alpha}_{w_{M_i}})} \prod_{t\in T} \mu_A(T) f_i'(v)(T) \quad (4.2)$$

Prior to defining the DPI for the class of fuzzy simple games we state andprove following results as follows.

Lemma 4.1. Given $A \in L(N)$ and let $w_M \in G^m_{FC}(N)$ with $v \in G^N$being the associatedsimple game of w_M. Two players $i, j \in N$ are α symmetric for w in A iff $i, j \in N$ aresymmetric for v in each B_{ij} where $B_{ij} = \{k \in N: \mu_B(i) < \alpha \; and \; \mu_B(j) < \alpha$.

Proof. Let $A \in L(N)$ and $w_M \in G^m_{FC}(N)$, the players $i, j \in N$be symmetric for w_M in A then for all $B \in L(U)$ with $\mu_B(i) < \alpha$ and $\mu_B(j) < \alpha$and all $I, J \in L(N)$ such that

$\mu_I(i) = \mu_J(j) > \alpha$, $\mu_I(j) = 0 \; when \; j \neq iand\mu_J(i) = 0$ when $i \neq j$, we have

$$w_M(B \cup I) = w_M(B \cup J)$$

$$\Rightarrow \sum_{T\in M(B\cup I)^{\alpha}_{w_M}} \prod_{t\in T} \mu_A(T)v(T) = \sum_{T\in M(B\cup J)^{\alpha}_{w_M}} \prod_{t\in T} \mu_A(T)v(T) \quad (4.3)$$

It is clear that some minimal winning coalitions are common for both $M(B \cup I)^{\alpha}_{w_M}$ and$M(B \cup J)^{\alpha}_{w_M}$). So after being cancellation these from both sides of(4.3) we get

$$\sum_{T\in M(B\cup I)^{\alpha}_{w_{M_i}}} \prod_{t\in T} \mu_A(T)v(T) = \sum_{T\in M(B\cup J)^{\alpha}_{w_{M_j}}} \prod_{t\in T} \mu_A(T)v(T)$$

Now following the fact that $\mu_I(i) = \mu_J(j)$ and

$$M(B \cup I)^{\alpha}_{w_{M_i}} = \{T \in M(B \cup I)^{\alpha}_{w_{M_i}} : \mu_T(j) = \mu_B(j) for \; all \; j \neq i \; and \; \mu_T(j) = \mu_I(i) \; \forall \; j = i\}$$
$$= T^i_{\alpha} \cup I_{\alpha}$$
$$M(B \cup J)^{\alpha}_{w_{M_j}} = \{T \in M(B \cup I)^{\alpha}_{w_{M_j}} : \mu_T(i) = \mu_B(i) for \; all \; j \neq i \; and \; \mu_T(i) = \mu_J(j) \; \forall \; j = i\}$$
$$= T^j_{\alpha} \cup J_{\alpha}$$

Hence

$$\sum_{T\in M(B\cup I)^{\alpha}_{w_{M_i}}} \prod_{t\in T} \mu_A(T) \left(v(T^i_{\alpha} \cup I_{\alpha}) - v(T^j_{\alpha} \cup J_{\alpha})\right)$$

$$\Rightarrow v(T^i_{\alpha} \cup I_{\alpha}) - v(T^j_{\alpha} \cup J_{\alpha}) = 0 \quad (4.4)$$

Since (4.4) holds for all $\alpha \in (0,1]$, so $i, j \in N$are symmetric for v.

Conversely, given $i, j \in N$are symmetric for v. It follows from the definition of w_M,

$$w_M(B \cup I) - w_M(B \cup J) = \sum_{T\in M(B\cup I)^{\alpha}_{w_M}} \prod_{t\in T} \mu_A(T)v(T) = \sum_{T\in M(B\cup J)^{\alpha}_{w_M}} \prod_{t\in T} \mu_A(T)v(T)$$

Using the fact that some minimal winning coalitions are common for both $M(B \cup I)^{\alpha}_{w_M}$and $M(B \cup J)^{\alpha}_{w_M}$and $\mu_I(i) = \mu_J(j)$, we have,

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$$w_M(B \cup I) - w_M(B \cup J) = \sum_{T \in M(B \cup I)^{\alpha}_{w_{M_i}}} \prod_{t \in T} \mu_A(T)(v(T^i_\alpha \cup I_\alpha) - v(T^j_\alpha \cup J_\alpha))$$

As $i, j \in N$are symmetric for $v$so we have,

$$v(T^i_\alpha \cup I_\alpha) - v(T^j_\alpha \cup J_\alpha) = 0 \quad \forall \alpha \in (0,1]$$

Hence,

$$w_M(B \cup I) = w_M(B \cup J)$$

Lemma 4.2. Given $A \in L(N)$ and $B \in L(N)$. Let $w_M \in G^m_{FC}(N)$ with $v \in G^N$being the associatedsimple game of w_M. Player $i \in N$is an α null player for $w_M$in A iff $i$is null for $v \in G^N$in each $T \in M(B \cup I)^{\alpha}_{w_{M_i}}$) with $\mu_B(j) > \alpha$when $j \neq i$, $\mu_B(j)$>α when $j = i$and $I \in L(N)$ such that $\mu_I(j)$<$\alpha$when $j \neq i$and $\mu_I(j)$>$\alpha$when $j = i$.

Proof. Let $i \in N$is an α null player for $w_M$in A . Then for all $B \in L(N)$ and $I \in L(N)$ such that $\mu_I(j)$<$\mu_B(j)$ when $j \neq i$ we have

$$w_M(B \cup I) = w_M(B)$$

$\Longrightarrow \sum_{T \in M((B \cup I)^{\alpha}_{w_M})} \prod_{t \in T} \mu_A(T) v(T) = \sum_{T \in M((B)^{\alpha}_{w_M})} \prod_{t \in T} \mu_A(T) v(T)$ (4.5)

Now it is clear that $M((B \cup I)^{\alpha}_{w_M}) \sqsupseteq M((B)^{\alpha}_{w_M})$Then after canceling common term fromboth sides of(4.5) we get

$$\sum_{T \in M((B \cup I)^{\alpha}_{w_{M_i}})} \prod_{t \in T} \mu_A(t) v(T) = 0$$

$$M(B \cup I)^{\alpha}_{w_{M_i}} = \{T \in M(B \cup I)^{\alpha}_{w_{M_i}} : \mu_T(j) = \mu_B(j) for\ all\ j \neq i\ and\ \mu_T(j) = \mu_I(i)\ \forall\ j = i\}$$
$$= T^i_\alpha \cup I_\alpha$$

$$\sum_{T^i_\alpha \cup I_\alpha \in M((B \cup I)^{\alpha}_{w_{M_i}})} \prod_{t \in T^i_\alpha \cup I_\alpha} \mu_A(t) v(T^i_\alpha \cup I_\alpha) = 0$$
$$\Longrightarrow v(T^i_\alpha \cup I_\alpha) = 0$$
$$\Rightarrow v(T^i_\alpha) = 0$$

$\Longrightarrow v(T^i_\alpha \cup I_\alpha) - v(T^i_\alpha) = 0$ (4.6)

As (4.6) holds for all αso $i$is a null player for v in each $T \in M((B \cup I)^{\alpha}_{w_M})$.

Conversely, $i$is a null player for $v$in each $T \in M((B \cup I)^{\alpha}_{w_M})$.

$$w_M(B \cup I) - w_M(B) = \sum_{T \in M((B \cup I)^{\alpha}_{w_M})} \prod_{t \in T} \mu_A(t) v(T) - \sum_{T \in M((B)^{\alpha}_{w_M})} \prod_{t \in T} \mu_A(t) v(T)$$

Now following the same argument as above we get

$w_M(B \cup I) - w_M(B) = \sum_{T^i_\alpha \cup I_\alpha \in M((B \cup I)^{\alpha}_{w_{M_i}})} \prod_{) t \in T^i_\alpha \cup I_\alpha} \mu_A(t) v(T^i_\alpha \cup I_\alpha)$ (4.7)

As i is null player for v so $v(T^i_\alpha \cup I_\alpha) = v(T^i_\alpha)$ for all $T^i_\alpha \cup I_\alpha \in M\left(\left(B \cup I\right)^{\alpha}_{w_{M_i}}\right)$

Hence

$$w_M(B \cup I) - w_M(B) = \sum_{T^i_\alpha \cup I_\alpha \in M((B \cup I)^{\alpha}_{w_{M_i}}))} \prod_{t \in T^i_\alpha \cup I_\alpha} \mu_A(t) v(T^i_\alpha)$$

Following (4.7) we get $v(T^i_\alpha)$= 0. So, $w_M(B \cup I) - w_M(B) = 0$. Hence proved.

B.FUZZY DPI ON $G^m_{FC}(N)$

Theorem 4.4. The function $f : G_{FC}(N) \rightarrow (\mathbb{R}^n)^{l(N)}$given by (4.2) is the Deegan-Packelindex for the class of fuzzy simple games in multilinear extension form given by(4.1) where$f'$is the associated crisp Deegan-Packel index.

In order to prove theorem(4.4) we need to show that the function given in(4.2) satisfiesAxiom F1-F4.

Proof. Axiom F1 (Efficiency): Let $A_\alpha = \{i_1, i_2, \ldots i_n\}$, where $m \leq n$.

$$f^{\alpha}_{i_k}(w_M)(A) = \sum_{T \in M\left(A^{\alpha}_{w_{M_{i_k}}}\right)} \prod_{i_j \in T} \mu_A(i_j) f'_{i_k}(v)(T)$$

$$\Longrightarrow \sum_{k=1}^{m} f^{\alpha}_{i_k}(w_M)(A) = \sum_{k=1}^{m} \sum_{T \in M(A^{\alpha}_{w_{M_{i_k}}})} \prod_{i_j \in T} \mu_A(i_j) f'_{i_k}(v)(T)$$

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$$= \sum_{T\in M(A^{\alpha}_{w_{M_{i_1}}})} \prod_{i_j\in T} \mu_A(i_j) f'_{i_1}(v)(T) + \cdots + \sum_{T\in M(A^{\alpha}_{w_{M_{i_m}}})} \prod_{i_j\in T} \mu_A(i_j) f'_{i_m}(v)(T)$$

Let, $M(A^{\alpha}_{w_M}) = \{T_1, T_2, \ldots, T_p\}$. Hence,

$$\sum_{k=1}^{m} f^{\alpha}_{i_k}(w_M)(A) = \prod_{i_j\in T_1} \mu_A(i_j) \sum_{i_j\in T_1} f'_{i_j}(v)(T_1) + \cdots + \prod_{i_j\in T_p} \mu_A(i_j) \sum_{i_j\in T_p} f'_{i_j}(v)(T_p)$$

Let, $T_1 = \{i_1, i_2, \ldots, i_l\}$, where, $l \le m$.Hence the cardinality of $T_1$is l. Hence following (2.9)

$$\sum_{i_j\in T_1} f'_{i_j}(v)(T_1) = \frac{1}{t_1} + \frac{1}{t_1} + \frac{1}{t_1} + \cdots + \frac{1}{t_1}(t_1\ times) = 1$$

Similarly,

$$\sum_{i_j\in T_p} f'_{i_j}(v)(T_p) = 1$$

Therefore,

$$\sum_{k=1}^{m} f^{\alpha}_{i_k}(w_M)(A) = \prod_{i_j\in T_1} \mu_A(i_j) + \cdots + \prod_{i_j\in T_p} \mu_A(i_j)$$
$$= w_M(A)$$

Axiom f2 (Null player Axiom): Null player follows from Lemma(4.2)
Axiom f3 (Symmetry): Symmetry follows from Lemma(4.1)
Axiom F4 (DP-minimal monotonocity): Let $w_M, w'_M \in G^m_{FC}(N)$. Given that for all$\alpha \in (0,1]$
$M(A^{\alpha}_{(w_m)_i}) \subseteq M(A^{\alpha}_{(w'_M)_i})$ (4.8)
Hence

$$|M(A^{\alpha}_{w_M})| \le |M(A^{\alpha}_{w'_M})|$$

Following (4.2) and due to (4.8) we get

$$f^{\alpha}_i(w'_M(A) \ge f^{\alpha}_i(w_M(A)$$

Therefore,

$$f^{\alpha}_i(w'_M(A) |M(A^{\alpha}_{w'_M})| \ge f^{\alpha}_i(w_M(A)|M(A^{\alpha}_{w_m})|$$

Let $N = \{1, 2, 3, 4\}$. Let us take an illustrative example where $v : 2^N \to [0, 1]$ is the simple game. Let Abe a fuzzy coalition over Ngiven by

$$A = \{< 1, 0.1 >, < 2, 0.2 >, < 3, 0.3 >, < 4, 0.4 >\}$$

Let, αbe 0.1, then

$$W(A^{\alpha}_{w_M}) = \{\{2, 4\}, \{3,4\}, \{2, 3,4\}\}$$

Hence

$$M(A^{\alpha}_{w_M}) = \{\{2, 4\}, \{3,4\}\}$$

Thus using (4.1), $w_M(A) = 0.2$. After some computation a Deegan Packel index, on $G^m_{FC}(N)$isobtained as $(0, 0.04, 0.06, 0.1)$.

VI. CONCLUSION

In this paper we introduce the notion of fuzzy simple games in multilinear extension form. Next we define Deegan-Packel index on this class of simple games. Deegan-Packel index what is defined here are generalization of [1, 7]. For the future work some other power indices may be considered.

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