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# A Non-Standard Method for the Solution of Some First Order Initial Value Problems

**A.A. Obayomi, M.O. Oke, A.E. Adenipekun, O.S. Adetolaju**

Department of Mathematics, Ekiti State University, Ado – Ekiti, Nigeria
Department of Mathematics, Ekiti State University, Ado – Ekiti, Nigeria
Department of Statistics, Federal Polytechnic, Ede, Osun State, Nigeria
Department of Computer Science, Ekiti State University, Ado – Ekiti, Nigeria

**ABSTRACT:** Finite difference schemes have been found to be very useful discrete models in the simulation and analysis of ordinary differential equations. Non-standard method is a family of techniques designed to provide a relatively simple way of creating discrete models that can correctly replicate the solution of physical applications that are modeled by linear and nonlinear differential equations. In this paper, we have used a combination of simple interpolating functions to derive some numerical models which are subjected to some transformations using rules 2 and 3 of the non-standard method. The resulting discrete models have been found to replicate the dynamics of the original equation and it shows some appreciable reduction in the absolute error of deviation when compared with the analytic solution.

**KEYWORDS:** Non-standard methods, Hybrid, Interpolation functions, Discrete model, Dynamical systems

## I. INTRODUCTION

Differential equation remains one of the foremost method of representing models that best describes a physical phenomenon. These are mathematical models which represent some dynamical state or behavior of such physical phenomena. However this differential model may or may not possess solution that can be explicitly expressed as a function of the unknown variables. In this case, simulation models become handy to analyse the state and status of such physical problems. Numerical methods have been in the forefront of creating such simulation models which we can use to experiment and predict the behaviour of the original physical phenomena.

Numerical approximations to the solutions of differential equations using finite difference methods are usually based on some acceptable rules and desirable qualities. Early numerical analysts are primarily concerned with standard issues like stability, convergence and consistency of the methods. The works of [1 and 2] are some of the widely referenced in this area. Numerical solutions have currently gone through diverse techniques and analysts has become more interested in numerical models that preserve the qualitative properties of the original differential equation and also behave consistently with reduced possibility of numerical instability.

Researchers have shown that most of the early standard schemes always produce solution curves that do not have the qualitative properties of the original dynamic equations [3 - 6]. The works of [3], [4] and [7 have laid a standard foundation for modeling using non-standard methods that can produce stable schemes that carried along the behavioral pattern of a dynamic system whose initial conditions are known. These later techniques are referred to as non-standard methods

## II. DERIVATION OF THE DISCRETE MODEL

Let us assume an interpolation function of the form

$$y(x) = a_1 x + a_2 x^2 \quad . \qquad (1)$$

$$y'(x) = a_1 + 2a_2 x \qquad (2)$$

$$y''(x) = 2a_2 \qquad (3)$$

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$$a_2 = \frac{y''(x)}{2} \quad (4)$$

From (2),

$$a_1 = y'(x) - 2a_2 x \quad (5)$$

Putting (4) in (5), we have

$$a_1 = y'(x) - xy''(x) \quad (6)$$

Let

$$y(x_n) = a_1 x_n + a_2 x_n^2$$

$$y(x_{n+1}) = a_1 x_{n+1} + a_2 x_{n+1}^2$$

$$y(x_{n+1}) - y(x_n) = a_1 x_{n+1} + a_2 x_{n+1}^2 - a_1 x_n - a_2 x_n^2$$

$$y(x_{n+1}) = y(x_n) + a_1(x_{n+1} - x_n) + a_2(x_{n+1}^2 - x_n^2) \quad (7)$$

Putting (4) and (6) in (7), we have

$$y(x_{n+1}) = y(x_n) + \{y'(x) - xy''(x)\}(x_{n+1} - x_n) + \frac{y''(x)}{2}(x_{n+1}^2 - x_n^2) \quad (8)$$

but

$x_n = a + nh$, $x_{n+1} = a + (n+1)h$, Therefore

$$x_{n+1} - x_n = h \text{ and } (x_{n+1}^2 - x_n^2) = 2ah + 2nh^2 + h^2 \quad (9)$$

putting (9) in (8), we have:

$$y(x_{n+1}) = y(x_n) + \{y'(x) - xy''(x)\}h + \frac{y''(x)}{2}(2ah + 2nh^2 + h^2) \quad (10)$$

Putting $y'(x) = f_n$ and $y''(x) = f_n'$ in (10), we have

$$y(x_{n+1}) = y(x_n) + (f_n - xf_n')h + \frac{f_n'}{2}(2ah + 2nh^2 + h^2)$$

Then,

$$y(x_{n+1}) = y(x_n) + hf_n - (a + nh)hf_n' + \frac{f_n'}{2}(2ah + 2nh^2 + h^2)$$

OR

$$y_{n+1} = y_n + hf_n - (a + nh)hf_n' + \frac{(2ah + 2nh^2 + h^2)}{2}f_n' \quad (11)$$

Equation (11) above is the standard numerical scheme. It is a one step method. The qualitative properties of the scheme are given in [1], [2], [8] and [9]. The proof of convergence, stability and consistency of the schemes are given in [9].

## III. NON-STANDARD SCHEMES

To derive the Non-standard schemes, we will apply rules 2 and 3 of [4] to each of the components of the equations. The rules are as stated below

**Rule 2**

Denominator function for the discrete derivatives must be expressed in terms of more complicated function of the step-sizes than those conventionally used. This rule allows the introduction of complex analytic function of h that satisfy certain conditions in the denominator.

**Rule 3**
The non-linear terms must in general be modeled (approximated) non-locally on the computational grid or lattice in many different ways.
Application of the combination of these two rules will give us the following transformations:

$\frac{dy}{dx} \equiv \frac{(y_{k+1}-y_k)}{\psi}$ where $\psi(h) \to h + 0(h^2)$ as $h \to 0$ (12)

The following non-local approximations for

$y_{k+1} \equiv ay_{k+1} + by_k \quad a + b = 1$ (13)

Sample renormalisation functions to be employed are

$\psi = \sin(\propto h)$ , $\propto \in R$ $\quad \to h + 0(h^2)$ as $h \to 0$ (14)

$\psi = \frac{(e^{\lambda h}-1)}{\lambda}$ , $\lambda \in R$ , $\quad \to h + 0(h^2)$ as $h \to 0$ (15)

## IV. APPLICATION TO SOME INITIAL VALUE PROBLEMS

**Problem 1**

Let us consider $(y')^2 = \frac{1-y^2}{1-x^2}$, $y(0) = \frac{\sqrt{3}}{2}$, [10]

To apply the scheme

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2} f_n'$$

$$y' = \sqrt[2]{\frac{1-y^2}{1-x^2}} = f_n$$

$$y'' = xy' - y = f_n'$$

$$y_{n+1} = y_n + h\left\{\sqrt[2]{\frac{1-y^2}{1-x^2}}\right\} + \frac{h^2}{2}\left\{x\left(\sqrt[2]{\frac{1-y^2}{1-x^2}}\right) - y\right\} \quad \ldots\ldots \text{ NEWh}$$

The two hybrid scheme will be obtained by changing h to $\psi = \sin(\propto h)$ and $\psi = \frac{(e^{\lambda h}-1)}{\lambda}$
which will be named NEWSin, NEWExp respectively
Applying the non-local approximation and transformation equations, we have

$$\left[\frac{y_{k+1}+y_k}{\psi}\right] = y' = \sqrt[2]{\frac{1-y_k^2}{1-x_k^2}}$$

$$y_{k+1} = y_k + \psi\left\{\sqrt[2]{\frac{1-y_k^2}{1-x_k^2}}\right\} ,$$

The three Nonstandard scheme is obtained by using $\psi = h$ NSh, $\psi = \sin(\delta h)$ and $\psi = \frac{(e^{\lambda h}-1)}{\lambda}$
which will be named NSh, NSsin, NSe respectively

**Problem 2**

$$yy' = -x , \quad y(0) = 5$$

$$y(x) = \begin{cases} -\sqrt[2]{25 - x^2} & 0 \le x < 5 \\ \sqrt[2]{25 - x^2} & -5 \le x < 0 \end{cases} , [10]$$

To apply the scheme

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2} f_n'$$

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$$y' = \frac{-x}{y} = f_n$$

$$y'' = \frac{-(x^2+y^2)}{y^3} = f_n'$$

$$y_{n+1} = y_n + h\left\{\frac{-x_k}{y}\right\} - \frac{h^2}{2}\left\{\frac{(x_k{}^2+y_k{}^2)}{y^3}\right\} \ldots\ldots \qquad \text{NEWh}$$

The two hybrid schemes will be obtained by changing h to $\psi = \sin(\propto h)$ and $\psi = \frac{(e^{\lambda h}-1)}{\lambda}$

which will be named NEWSin, NEWExp respectively

Applying non-local approximation and transformation equations, We have the following

$$\left[\frac{y_{k+1}+y_k}{\psi}\right] = y' = \frac{-x_k}{y_k}$$

$$y_{k+1} = y_k - \psi\left\{\frac{x_k}{y_k}\right\},$$

The three Nonstandard scheme is obtained by using $\psi = h$ NSh, $\psi = \sin(\delta h)$ and $\psi = \frac{(e^{\lambda h}-1)}{\lambda}$ which will be named NSh, NSsin, NSe respectively

## V. GRAPHICAL REPRESENTATION OF THE NUMERICAL RESULTS

The schemes developed for the two initial value problems considered have been coded into a numerical solver and applied using various parameters. The results of the numerical experiment are presented in 3D graphs below.

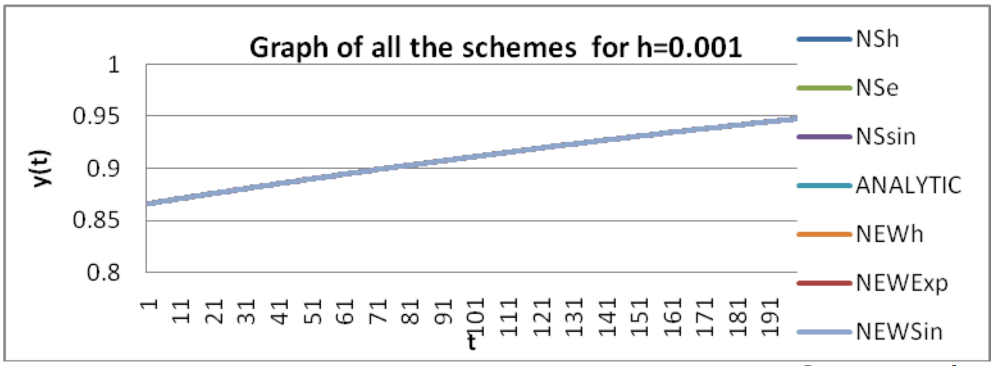


**Fig 1: Graph of Schemes and the analytic solution for** $(y')^2 = \frac{1-y^2}{1-x^2}, y(0) = \frac{\sqrt{3}}{2}$

The simulated experiment produced solution curves which have the same monotonicity as the curves of the analytic solution. There are no extraneous solutions.

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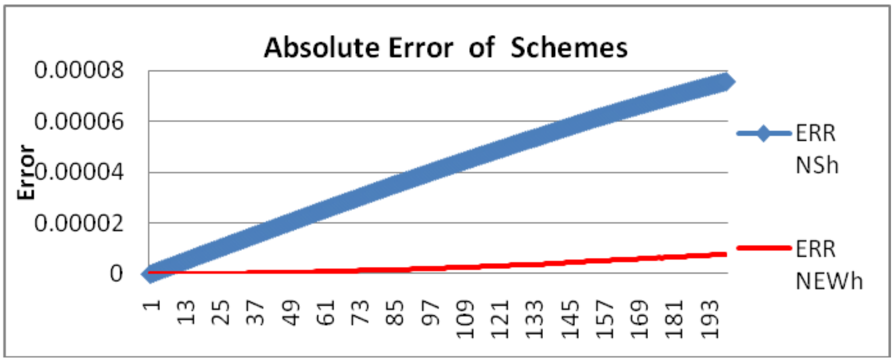


**Fig 2a: Absolute error of deviation of the standard schemes from the analytic solution using h =0.001**

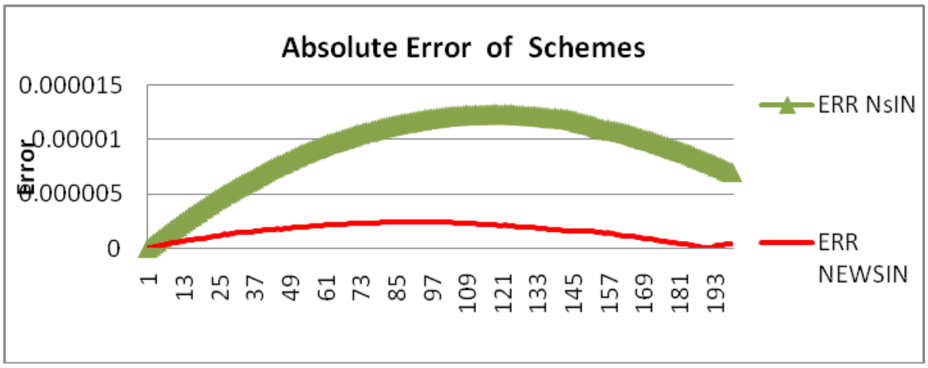


**Fig 2b: Absolute error of deviation of the Non-standard schemes from the analytic solution using h =0.001**


**Fig 2c: Absolute error of deviation of the Non-standard schemes from the analytic solution using h =0.001**

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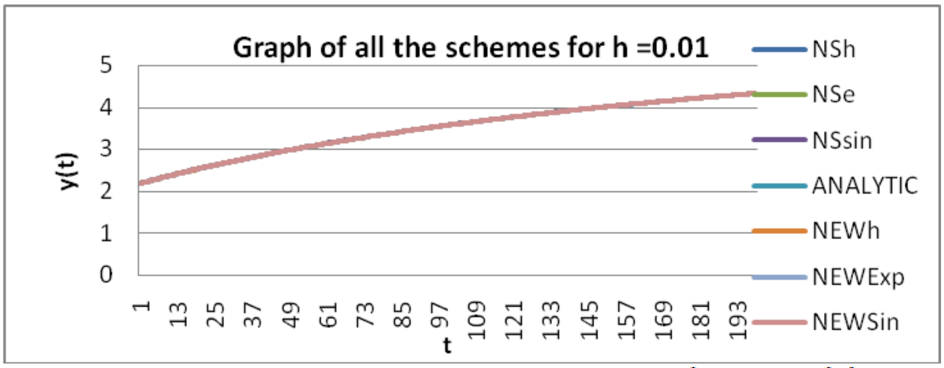


**Fig 3: Graph of Schemes and the analytic solution for** $yy' = -x$ , $y(0) = 5$

The schemes produced solution curves which have the same monotonicity as the curves of the analytic solution. There are no extraneous solutions. These curves present the solution in the positive region

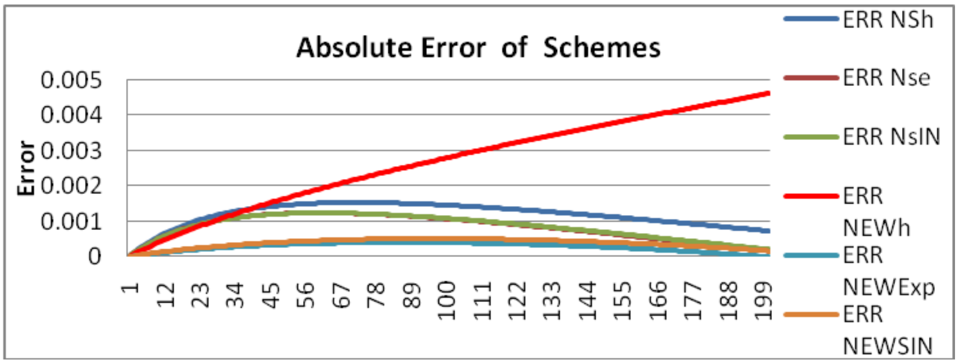


**Fig 4: Absolute error of deviation of all the schemes from the analytic solution using h =0.01**

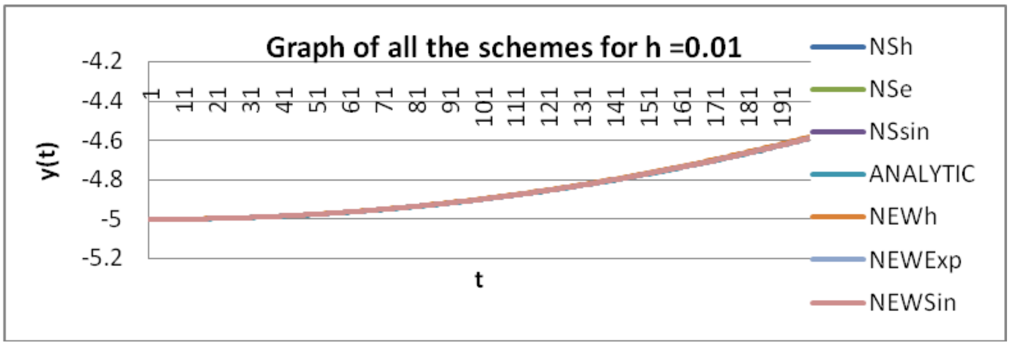


**Fig 5: Graph of Schemes and the analytic solution for** $yy' = -x, y(0) = -5$

The simulated experiment produced solution curves which have the same monotonicity as the curves of the analytic solution. There are no extraneous solutions. These curves represent the solution in the negative region.

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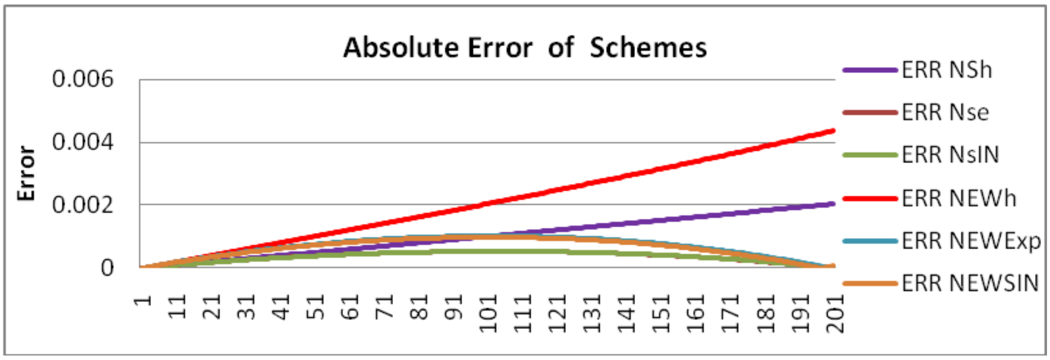


**Fig 6: Absolute error of deviation of all the schemes from the analytic solution using h =0.01**

## VI. ERROR ANALYSIS

The absolute error of deviation for each scheme is depicted in Figures 2a, 2b, 2c, 4 and 6 for 200 iterations respectively. The absolute error for the Non-standard schemes NSsin, Nse, NewSin and NEWExp remain below 0.001 and this is consistent and proportional for any step-size.
The standard scheme NEWh produced an absolute deviation value that is higher than the Non-standard ones.

## VII. SUMMARY AND CONCLUSION

It will be observed that the Standard schemes NSh and NEWh are central difference schemes with the standard denominator which is the step-size. These two schemes are also suitable for the simulation of the differential equation. All the other schemes have been put through some form of Non-standard transformations. The combination of the two techniques for the hybrid schemes produced better solution curves with lower absolute deviation from the analytic solution as we can see in figures 2, 4 and 6. The new schemes have been shown to posses desirable qualitative properties. From the results of these experiments on the sampled equations it can be seen that all the schemes are stable with respect to monotonicity of solution i.e the orbit of the schemes behave exactly like the orbit of the original equation as we can see in figures 2, 4 and 6.

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