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On Weak ** Commutativity and Rotativity Conditions of Mappings in Common Fixed Point Considerations

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ABSTRACT: The concept of weak* commuting mappings was given by H.K. Pathak [3]. has generalized some results of B. Fisher [2] on fixed point theorem by using the concept to weak ** commuting mapping. We have two common fixed point theorems for three self maps of a complete metric space satisfying a rational inequality by using the concepts of weak ** commuting maps and rotativity of maps. We further extend the results of Diviccaro, Sessa and Fisher [1].

KEYWORD: Weak ** commuting mapping, Rotativity of maps, Complete metric space.

I. INTRODUCTION

We begin with the following known definitions:-

Definition 1 : Let (X,d) be a space and let S and I be mappings of X in to itself. We define the pair (S,I) to be weak ** commuting.

if $S(X)\subset I(X)$

and $d(S^2I^2x, I^2S^2X) \leq d(S^2Ix, IS^2x) \leq d(SI^2x, I^2Sx) \leq d(SIx, ISx) \leq d(S^2x, I^2)$

for all x in X.

It is obvious that two commuting mapping are also weak ** commuting, but two weak**commuting do not necessarily commute as shown in example 1 below.

Definition 2 : A map $T:X\rightarrow X$ is called idempotent, if $T^2 = T$. We note that if mappings are idempotent, then our definition of weak ** commuting of pair (S,I) reduces to weak commuting of pair (S,I) defined by Sessa [5].

Definition 3 : The map T is called rotative w.r.t.I, If $d(Tx, I^2x) \leq d(Ix, T^2x)$

for all x in X. clearly if T and I are idempotent maps, then definition is obvious.

Common fixed point theorems for a weak ** commuting pair of mappings.

In this section, we have some results on common fixed points for three self maps of a complete metric space satisfying a rational inequality by using the concepts of weak ** commuting maps and rotativity of maps. The following theorem generalizes the result of Diviccaro, Sessa and fisher [1]

Theorem 1. Let S, T and I be three mappings of a complete metric space (X,d) such that foa all x, y in X either

(I) $d(S^2x, T^2y) \leq K' [d(I^2x, S^2x) + d(I^2y, T^2y)]$

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$$+K\frac{[d(I^2x,S^2x).d(I^2y,T^2y)+d(I^2x,T^2y).d(I^2y,S^2x)]}{d(I^2x,S^2x)+d(I^2y,T^2y)}$$

if $d(I^2x, S^2x)+ d(I^2y, T^2y) \neq 0$, where $K' < 1$, and $(K+K')<1/2$, or

(II) $d(S^2x, T^2y) = 0$ if $d(I^2x, S^2x)+ d(I^2y, T^2y)=0$

Suppose that the range of I^2 contains the range of S^2 and T^2. If either

(a_1) I^2 is continuous, I is weak ** commuting with S and T is rotative w.r.t. I,

(a_2) I^2 is continuous, I is weak ** commuting with T and S is rotative w.r.t. I,

(a_3) S^2 is continuous, S is weak ** commuting with I and T is rotative w.r.t. S,

(a_4) T^2 is continuous, T is weak ** commuting with I and S is rotative w.r.t. T

Then S, T and I have a unique common fixed point z. Further, z is the unique common fixed point of S and I and T and I.

Proof. Let x_0 be an arbitrary point in X. Since the range of I^2 contains the range of S^2, let x_1 be a point in X such that $S^2 x_0 = T^2x_1$. Since the range of I^2 contains the range of T^2, we can choose a point x_2 such that $T^2x_1 = I^2x_2$ in general, having chosen the point x_{2n} such that :

$$S^2x_{2n} = I^2x_{2n+1}$$

and $$T^2x_{2n+1} = I^2x_{2n+2} \quad \text{for } n = 0,1,2 \ldots\ldots\ldots\ldots$$

Now we distinguish three cases :

Case I. Let $d_{2n-1} \neq 0$ and $d_{2n} \neq 0$ for n = 1,2 .... then, We have

$d_{2n-1} + d_{2n} = d(I^2 x_{2n}, S^2 x_{2n}) + d(I^2 x_{2n+1}, T^2x_{2n+1}) \neq 0$, for n = 1,2..........

Using inequality (I), we then have

$d_{2n} = d(S^2x_{2n}, T^2x_{2n+1})$

$\leq K(d_{2n-1}+d_{2n}) + K.\left[\frac{d_{2n-1}\ d_{2n}+d(T^2 x_{2n-1}T^2x_{2n+1}).d(S^2x_{2n,}S^2x_{2n})}{d_{2n-1}+d_{2n}}\right]$

ie., $d_{2n} \leq K(d_{2n-1}+d_{2n}) + K\left[\frac{d_{2n-1}.\ d_{2n}}{d_{2n-1}+d_{2n}}\right]$

ie., $d_{2n} \leq K(d_{2n-1}+d_{2n}) + K(d_{2n-1}+d_{2n})$

Then $d_{2n} \leq \frac{(K'+K)}{(1-K'-K)} d_{2n-1}$

which implies that

$d_{2n} < d_{2n-1}$ since $(K'+K')< ½$

Then

(1) $d(S^2x_{2n-1}, T^2x_{2n+1}) < d(T^2x_{2n-1,}S^2x_{2n})$ for n = 1,2............

Similarly, it is proved that $d_{2n-1}< d_{2n-2}$

So $d(T^2x_{2n-1,}S^2x_{2n}) < d(S^2x_{2n-1,}T^2x_{2n-1})$ for n = 1,2............

It follows that the sequence

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(2) $\{ S^2x_0, T^2x_1, S^2x_2 .... T^2x_{2n-1}, S^2x_{2n}, T^2x_{2n+1} ....\}$

is a Cauchy sequence in the complete metric space X and so has a limit w in X.

Hence the sequence

$$\{S^2x_{2n}\}= \{I^2x_{2n-1}\} \text{ and } \{ T^2x_{2n-1}\} = \{I^2x_{2n}\}$$

converge to the point w because they are subsequences of the sequence (2). Suppose first of all that I^2 is continuous, then the sequence $\{I^4x_{2n}\}$ and $\{I^2S^2x_{2n}\}$ converge to the point I^2 w. If I weak ** commutes with S, we have

$$d(S^2I^2x_{2n}, I^2w) \le d(S^2I^2x_{2n}, I^2S^2x_{2n}) + d(I^2S^2x_{2n}, I^2w)$$

$$\le d(S^2x_{2n}, I^2x_{2n}) + d(I^2S^2x_{2n}, I^2w)$$

which implies , on letting n tend to infinity that the sequence $\{S^2I^2x_{2n}\}$ also converges to I^2w. We now claim that $T^2w=I^2w$. Suppose not. Then we have $d(I^2w,T^2w)> O$ and using inequality (I), we obtain

$$d(S^2I^2x_{2n},T^2w) \le K'[d(I^4x_{2n}, S^2I^2x_{2n})+d(I^2w,T^2w)]$$

$$+ K\left[\frac{d(I^4x_{2n},S^2I^2x_{2n}).d(I^2w,T^2w)+d(I^4x_{2n},T^2w).d(T^2w,S^2I^2x_{2n})}{d(I^4x_{2n},S^2I^2x_{2n})+d(I^2w,T^2w)}\right]$$

On letting n tend to infinity, we deduce that

$$d(I^2w,T^2w) \le K'.d(I^2w,T^2w)$$

i.e $(1-K')\, d(I^2w,T^2w) \le 0$ a contradiction since $K'<1$.

Now suppose that $S^2w \neq T^2w$, then

$$d(S^2w,T^2w) \le K' [d(I^2w, S^2w) + d(I^2w,T^2w)]$$

$$+K\left[\frac{d(I^2w,S^2w).d(I^2w,T^2w)+d(I^2w,S^2w).d(I^2w,T^2w)}{d(I^2w,S^2w)+d(I^2w,T^2w)}\right]$$

i.e. $d(S^2w,T^2w) \le K'd(T^2w,S^2w)$

i.e. $(1-K')\, d(T^2w, S^2w) < 0$ a contradiction.

Thus $I^2w = S^2w = T^2w$.

A similar conclusion is achieved if I weak ** commute with T. Let us now suppose that S^2 is continuous instead of I^2. The in subsequences $\{S^4x_{2n}\}$ and $\{S^2I^2x_{2n}\}$ converge to the point S^2w. Since S weak ** commutes with I, we have that the sequence $\{I^2S^2x_{2n}\}$ also converges to S^2w. Since the range I^2 contains the range of S^2, there exists a point w', such that

$$I^2w' = S^2w$$

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Then $T^2w \neq S^2w = I^2w'$, we have

$d(S^4x_{2n}, T^2w') \leq K'[d(I^2S^2x_{2n}, S^4x_{2n}) + d(I^2w', T^2w')]$

$+ K[\frac{d(I^2S^2x_{2n},S^4x_{2n}).d(I^2w',T^2w)+d(I^2S^2x_{2n},T^2w)'.d(I^2w',S^4x_{2n})}{d(I^2S^2x_{2n},S^4x_{2n}).d(I^2w',T^2w')}]$

and on letting n tend to infinity, it follows that

$d(S^2w, T^2w') \leq K' [d(S^2w, S^2w) + d(I^2w', T^2w')]$

$+ K [\frac{d(S^2w,S^2w)).d(I^2w',T^2w')+d(S^2w,T^2w').d(I^2w',S^2w)}{d(S^2w,S^2w)).d(I^2w,T^2w')}]$

i.e. $d(S^2w,T^2w') \leq K' d(S^2w,S^2w')$

i.e. $(1-K').d(S^2w,T^2w') \leq 0$, which is a contradiction.

Thus $S^2w= T^2w' = I^2w'$. Now suppose that $S^2w \neq T^2w = I^2w'$,

Then $d(S^2w, T^2w')$

$\leq K' [d(S^2w',S^2w') + d(I^2w',T^2w')]$

$+ K [\frac{d(S^2w',S^2w').d(I^2w',T^2w')+d(I^2w', T^2w').d(I^2w',S^2w')}{d(S^2w',S^2w')).d(I^2w',T^2w')}]$

$= 0$, a contradiction, and so $I^2w' = S^2w' =T^2w'$

A similar conclusion is obtained if one assumes that T^2 is continuous and T is weak ** commuting with I.

Cass II. Let $d_{2n-1} = 0$ for some n. Then $I^2x_{2n} = T^2x_{2n-1} = S^2x_{2n}$.

We claim $I^2x_{2n} = T^2x_{2n}$, since otherwise

if $d(I^2x_{2n}, T^2x_{2n}) > 0$, inequality (I) implies,

$0 < d(I^2x_{2n}, T^2x_{2n}) = d(S^2x_{2n}, T^2x_{2n})$

$\leq K'[d(I^2x_{2n}, S^2x_{2n}) + d((I^2x_{2n}, T^2x_{2n})]$

$$+K[\frac{d(I^2X_{2n},S^2X_{2n})d(I^2X_{2n},T^2X_{2n})+d(I^2X_{2n},T^2X_{2n}).d(I^2X_{2n},S^2X_{2n})}{d(I^2X_{2n},S^2X_{2n})+d((I^2X_{2n},T^2X_{2n})}]$$

$= K'[d_{2n-1}+ d((I^2x_{2n}, T^2x_{2n})]$

$$+K[\frac{d_{2n-1}.d(I^2x_{2n},T^2x_{2n}) + d(I^2x_{2n},T^2x_{2n}).d_{2n-1}}{d_{2n-1} + d(I^2x_{2n},T^2x_{2n})}]$$

i.e. $0 < d (I^2x_{2n}, T^2x_{2n}) \leq K' .d(I^2x_{2n}, T^2x_{2n})$

i.e. $0 < d (1-K') .d(I^2x_{2n}, T^2x_{2n}) \leq 0$, a contradiction.

Thus $I^2x_{2n} = S^2x_{2n}=T^2x_{2n}$.

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Case III. Let $d_{2n}=0$ for some n. Then $I^2x_{2n+1}= S^2x_{2n} = T^2x_{2n+1}$ and reasoning as in Case(II) $I^2x_{2n+1}=S^2x_{2n+1}=T^2x_{2n+1'}$

Therefore in all cases , there exists a point w such that $I^2w=S^2w=T^2w$.

If I weak ** commutes with S, we have

$d(S^2Iw,IS^2w)\leq d(SI^2w,I^2Sw)\leq d(SIw, ISw)\leq d(S^2w,I^2w)=0$, which implies that

(3) $S^2Iw=IS^2w$, $SI^2w=I^2Sw$, $SIw= ISw$, and so $I^2Sw=S^3w$.

Thus $d(I^2Sw, S^2Sw) + d(I^2w,T^2w) = 0$ and using condition (II), we deduce that $SI^2w = S^2Sw =T^2w =I^2w$.

It follows that $I^2 w = z$ is fixed point of S.

Further $d(I^2Iw, S^2Iw) + d(I^2w,T^2w) = 0$

and using condition (II), we deduce that $Iz = S^2Iw = IS^2w = T^2w = z$ and using inequality (I), on the assumption that $T^2z \neq z$, we have

$d(z,T^2z) = d(S^2z,T^2z)$

$\leq K' [d(I^2z,S^2z) + d(I^2z, T^2z)]$

$+K[\frac{d(I^2z,S^2z).d(I^2z,T^2z)+d(I^2z,T^2z).d(I^2z,S^2z)}{d(I^2z,S^2z)+d(I^2z,T^2z)}]$

i.e., $d(z,T^2z) \leq K'.d(z,T^2z)$

i.e., $(1-K') d(z,T^2z) < 0$, a contradiction.

And so $z = T^2z$.

Now using the rotativity of T w.r. to I (or w.r. to S), we have

$d(Tz,z)= d(Tz,I^2z) \leq d(Iz,T^2z) = d(z,z) = 0$,

and so z is a common fixed point of I, S and T.

If one assumes that I weak ** commutes with T and S is rotativity w.r. to I (or w.r. to T), the proof is of course similar.

Now suppose that z' is a second common fixed point of I and S. Then

$d(I^2z',S^2z') + d(I^2z, T^2z) = 0$ and condition (II) implies that

$z' = Sz' = S^2z' = T^2z = z$.

We can prove similarly that z is the unique common fixed point of I and T.

This completes the proof of the theorem.

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Example 1.

Let X be the subset of R^2 defined by

X = (A,B,C,D,E),

where A ≡ (0,0), B ≡ (0,1), C ≡ (0, 1), D = (1/2,0), E ≡ (- 1,0).
Let I, S, T : X → X be given by

I(A) = I(B) = I(C) = B, I(D) = A, I(E) = D,
S(A) = S(B) = S(C) = B, S(D) = S(E) = A,

T(A) = T(B) = T(C) = T(D) = T(E) = B.

By routine calculation it is easy to see that I weak ** commutes with S and T is rotative w.r.to S. Clearly I^2 (or S^2) is continuous and

$S^2(X)=\{B\} \subset \{A,B\} = I^2(X)$ and $T^2(X) =\{B\}\subset\{A,B\} = I^2(X)$.

Further, and easy routine calculation shows that inequality (I) holds for instance K' < 1, and (K + K') < 1/2 and condition (II) holds for the points x, y ∈ {A,B,C,D}.

Therefore all the conditions of Theorem 1 are satisfied and B is the unique common fixed point of I, S and T.

We also note that is neither commutative nor weakly commutative with S, for otherwise,

SI(E) = A ≠ B = IS(E)

and d(SI(E),IS (E))= d(A,B) = 1 > 1/2 = d(A,D)

= d (S(E), I(E)).

Example 2.

Let X = {x,y} with the discrete metric. Define the mappings

I = S = T by Ix = x, Iy = y.

All the conditions of the Theorem 1 are satisfied except condition (II) but I, S and T. have two common fixed points.

Assuming $I = I^2$ (identity map on X) and dropping the rotativity of T(or S) we have the following corollary.

Corollary 2.

Let S and T be mappings of a complete metric space (X,d) into itself such that for all x,y in X either,

(III) $d(S^2x,T^2y)$

$\leq K' [d(x,S^2x) + d(y,T^2y)]$

$+ K[\frac{d(x,S^2x).d(y,T^2y)+d(x,T^2y).d(y,S^2x)}{d(x,S^2x)+d(y.T^2y)}]$

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If $d(x,S^2x) + d(y,T^2y) \neq 0$ where $K' < 1$ and $(K+K') < 1/2$,

or $d(S^2x,T^2y) = 0$ If $d(x,S^2x) + d(y,T^2y) = 0$

Then S and T have a unique common fixed point z. Further, z is the unique fixed point of S and of T.

Proof. It is not very hard to show that there exits a point $w \in X$ such that $w = S^2w = T^2w$.

Thus $d(Sw,S^2Sw) + d(w,T^2w) = 0$ and using condition (III), we deduce that $Sw = S^2Sw = T^2w = w$. Again $d(w,S^2w) + d(Tw,T^2Tw) = 0$ and so using condition (III), we deduce that $Tw = T^2Tw = Sw = w$. It follows that w is a common fixed point of S and T. The unicity of w follows easily. This completes the proof.

Remark 1.

If follows from the proof of the Theorem 1 that if condition (II) is omitted in the statement of Theorem 1 we can say that w is a concidence point of I^2, S^2 and T^2.

Remark 2.

Assuming I, S and T as idempotent maps of X, and K'=0, we obtain Theorem 1 of [1].

Remark 3.

Assuming I as identity map and S and T as idempotent map of X and K'=0, we obtain Theorem 3 of [2].

Remark 4.

Assuming I, S and T as idempotent maps of X and S=T on X, and K' =0, we obtain Corollary 2 of [1].

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