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A Constrained Q - G Programming Problem and Its Application

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ABSTRACT: In practice there exist many methods to solve unconstrained, constrained and mixed quadratic and geometric programming problems. In this paper an attempt is made to develop constraint Q – G programming problem by combining constrained quadratic programming problem and constrained geometric programming problem. This model is solved by using the technique of quadratic programming problem. A hypothetical example is considered to illustrate the model.

KEYWORDS: Constrained Quadratic Programming problems, Geometric Programming problem, Khun – Tucker conditions, Concave curve.

I INTRODUCTION

In practice various techniques are available to solve constrained quadratic problem. The objective of quadratic programming is to maximize or to minimize the quadratic objective function. Let decision variables $\underline{x}$ and the coefficients of objective function, $\underline{C} \in R^n$ and D be symmetric matrix of real numbers of order $n \times n$ then the constrained quadratic programming problem is define as follows:

$$Maximize \quad f(x) = \underline{C}' \cdot \underline{x} + \frac{1}{2}\underline{x}' \cdot D \cdot \underline{x}$$

$$Subject\ to\ the\ \ constraints$$

$$Q\underline{x} \le \underline{b}$$

$$and \quad \underline{x} \ge \underline{0}$$

Here $\underline{x}' \cdot D \cdot \underline{x}$ is in quadratic form and $D = (d_{ij})_{n \times n}$ is a symmetric matrix also $\underline{b} \in R^m$ and Q is a real matrix of order $m \times n$.

Geometric programming is a technique for solving a special case of nonlinear problems. Duffin, Peterson and Zener [2] published a book "*Geometric Programming: Theory and Applications*" that started the field of Geometric Programming as a branch of nonlinear optimization with many useful theoretical and computational properties of Geometric Programming, to a large extent the scope of linear programming applications and is naturally applied to several important nonlinear systems in science and engineering. Several important developments of Geometric Programming are in the area of mechanical and civil engineering, chemical engineering, probability and statistics, finance and economics, control theory, circuit design, information technology, coding and signal processing, wireless networking, etc. took place in the late 1960s to early 1970s. There are several books on nonlinear optimization that have a section on Geometric Programming, e.g., M. Avriel, [5], C. S. Beightler [1], G. Hadley [4], Taha [6], etc. However, many researchers felt that most of the theoretical, algorithmic and application aspects of Geometric Programming had been exhausted by the early 1980's, the period of 1980–98 was relatively quiet. After the revolution in the electronic field, over the last few years, Geometric Programming started to receive renewed attention from the operations research community.

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The constrained Geometric Programming in the following manner:

$$Min\ \ f(\underline{x}) = \sum_{k=1}^{N} C_k \prod_{i=1}^{n} x_i^{a_{ik}}$$

$$Subject\ to\ constraints$$

$$Q\left(\frac{1}{\underline{x}}\right) \leq \underline{b}$$

$$\underline{x} \geq \underline{0} \qquad [1]$$

Here it is assumed that the coefficient $c_k > 0$ and N is finite. The exponents a_{ik} are unrestricted in sign. $Q = (q_{li})$ is a $m \times n$ real matrix and $\underline{b} = (b_i)$ is a $m \times 1$ is constant.

Here, a model is considered in which the concept of quadratic programming problem and geometric programming problem is combined and hence the new model is defined as Q – G programming problem.

II ASSUMPTIONS

In the present study following assumptions are made to derive a solution to the constrained quadratic-geometric programming.

1. The coefficients are unrestricted in sign i.e. $c_i \geq 0\ \ or\ \ c_i \leq 0\ \ ;\ i = 1, 2, ...n$
2. N is finite i.e. number of terms is finite.
3. The number of terms $N = n + 1$ where n is number of variables.
4. $D = (d_{ij})_{n \times n}$ is a symmetric matrix.
5. The quadratic form of the model is negative semi definite.

III MATHEMATICAL MODEL AND PROCEDURE

Let $\left(\frac{1}{\underline{x}}\right)$ and $\underline{C} \in R^n$ and D is any real $n \times n$ symmetric matrix then constrained Q - G programming problem is defined as under:

$$Minimise\ \ \ Z = f(\underline{x}) = \underline{C}^{'}\left(\frac{1}{\underline{x}}\right) + \frac{1}{2}\left(\frac{1}{\underline{x}}\right)^{'} \cdot D \cdot \left(\frac{1}{\underline{x}}\right)$$

$$Subject\ \ to\ \ the\ constrints$$

$$A\left(\frac{1}{\underline{x}}\right) \leq \underline{b}$$

$$and \qquad \underline{x} > \underline{0} \qquad [2]$$

Or,

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$$Minimise\ Z = \sum_{i=1}^{n} \frac{c_i}{x_i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{d_{ij}}{x_i x_j}$$

subject to the constraints

$$\sum_{i=1}^{n} \frac{a_{li}}{x_i} \le b_l \qquad l = 1, 2, ..., m$$

$$and \qquad x_i > 0. \ \ i = 1, 2, ...n$$

The above problem can be converted in to constrained quadratic programming problem by taking $\frac{1}{\underline{x}} = \underline{y}\ or\ \frac{1}{x_i} = y_i \quad i = 1, 2, ..., n$ Hence the problem becomes

$$Maximise\ Z' = f(y_1, y_2, ..., y_n) = \sum_{i=1}^{n} c_i y_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} y_i y_j$$

subject to the constraints

$$\sum_{i=1}^{n} a_{li} y_i \le b_l \qquad l = 1, 2, ..., m$$

$$and \qquad y_i > 0. \ \ i = 1, 2, ...n$$

Now the above problem can be solved by Wolfe's method as follow:

Step 1: Convert the inequalities into equalities by introducing slack variables $s_l^2,\ l = 1, 2, ..., m$ and the slack variables $s_{m+i}^2,\ i = 1, 2, ..., n$ in the i^{th} non-negativity constraints

Step 2: Construct the Lagrangian function,

$$L(\underline{y}, \underline{s}, \underline{\lambda}) = f(y_1, y_2, ..., y_n) - \sum_{l=1}^{m} \lambda_l \left[\sum_{i=1}^{n} a_{li} y_i - b_l + s_l^2 \right] - \sum_{i=1}^{n} \lambda_{m+i} \left[-y_i + s_{m+i}^2 \right]$$

Where

$$\underline{y}' = (y_1, y_2, ..., y_n),$$

$$\underline{s}' = (s_1^2, s_2^2, ..., s_{m+n}^2),$$

$$\underline{\lambda}' = (\lambda_1, \lambda_2, ..., \lambda_n)$$

Step 3: Differentiate $L(\underline{y}, \underline{s}, \underline{\lambda})$ partially with respect to the components $\underline{y}$, $\underline{s}$, $\underline{\lambda}$ and equate the first order partial derivatives with equal to zero. Derive the Kuhn-Tucker conditions from the resulting equations as follows:

$$\sum_{j=1}^{n} d_{ij} y_j - \sum_{l=1}^{m} \lambda_l a_{li} + \lambda_{m+i} = -C_i \ (i = 1, 2, ..., n)$$

$$\sum_{i=1}^{n} a_{lj} y_i + S_l^2 = b_l \qquad (l = 1, 2, ...m)$$

$$\sum_{l=1}^{m} S_l^2 \lambda_l + \sum_{i=1}^{n} y_i \lambda_{m+i} = 0$$

$$\lambda_l, \lambda_{m+i}, \lambda_i \ge 0 \quad (l = 1, 2, ..., m), (i = 1, 2, ..., n)$$

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Step 4: Introduce the non-negative artificial variables $w_i, i=1,2,...,n$ in the above Kuhn-Tucker conditions

$$C_i + \sum_{j=1}^{n} d_{ij} y_j - \sum_{l=1}^{m} \lambda_l a_{li} + \lambda_{m+i} + w_i = 0 \;\; For \;\; (i=1,2,...,n)$$

$$Z = w_1 + w_2 + ... + w_n$$

Step 5: Obtain an initial basic feasible solution to the linear programming problem

$$Maximise\, Z' = w_1 + w_2 + ... + w_n$$

$$Subject \;\; to \;\; the \;\; constraints$$

$$\sum_{j=1}^{n} d_{ij} y_j - \sum_{l=1}^{m} \lambda_l a_{li} + \lambda_{m+i} + w_i - C_i = 0 \;\; For \;\; (i=1,2,...,n)$$

$$\sum_{i=1}^{n} a_{li} y_i + s_{n+l}^* = b_l \;\; (l=1,2,...,m)$$

$$w_i, \lambda_l, \lambda_{m+i}, y_i > 0, (l=1,2,...,m), (i=1,2,...,n)$$

$$where, \; s_{n+i}^* = s_l^2 \;\; (l=1,2,...,m)$$

and satisfying the complementary slackness condition:

$$\sum_{i=1}^{n} \lambda_{m+i} y_i + \sum_{l=1}^{m} s_{n+l}^* \lambda_l = 0$$

Step 6: Use two phase simplex method to obtain an optimum solution satisfying the complementary slackness condition to the linear programming problem obtained in step 5.

Step 7: The optimum solution obtained in step 6 is an optimum solution to the obtained quadratic programming problem also.

Step 8: Using optimum solution of y_i obtained in step 7 are converted into the optimum solution x_i by using the relation $x_i = \dfrac{1}{y_i}, \;\; i=1,2,3,....n$ which will minimize the given objective function of Q – G programming problem.

In this method at each iteration there is a basic solution containing $m+n$ variables. A basic solution at any iteration may correspond to one of the following two cases:

Case 1: For each l and each i the basis contains only one complementary variable, such a basic solution is called a standard basic solution and satisfies the complementary slackness constraints.

Case 2: For each l and each i the basis contains a basic pair of complementary variable, such basic solution is called nonstandard basic solution and may not satisfies the complementary slackness constraints.

The following principal modification is the back-bone of Wolfe's method:

"Whenever a nonstandard basis occurs the selection procedure for entering a new variable into the basis seeks to reestablish the complementary slackness condition"

IV.HYPOTHETICAL PROBLEM

Solve the following problem of Q - G Programming

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$$Minimise \quad Z = \frac{2}{x_1} + \frac{3}{x_2} - \frac{2}{x_1^2}$$

$$Subject \ to \ the \ contrants$$

$$\frac{1}{x_1} + \frac{4}{x_2} \le 4$$

$$\frac{1}{x_1} + \frac{1}{x_2} \le 2$$

$$x_1, x_2 > 0$$

The above problem can be solved as under

The above Q – G programming problem can be converted into quadratic programming problem by taking $y_i = \frac{1}{x_i}, \ i = 1, 2$ hence the above problem can be re-written as follows

$$Maximise \quad Z^* = 2y_1 + 3y_2 - 2y_1^2$$

$$Subject \ to \ the \ contrants$$

$$y_1 + 4y_2 \le 4$$

$$y_1 + y_2 \le 2$$

$$y_1, y_2 > 0$$

Now, converting inequalities into the equalities by introducing slack variables s_1^2 and s_2^2 respectively. Considering $y_1 > 0$ and $y_2 > 0$ also as inequality constraints and converting those also into the equalities by introducing another two slack variables s_1^2 and s_2^2 then the above problem can be written as

$$Maximise \quad Z' = 2y_1 + 3y_2 - 2y_1^2$$

$$Subject \ to \ the \ contrants$$

$$y_1 + 4y_2 + s_1^2 = 4$$

$$y_1 + y_2 + s_2^2 = 2$$

$$-y_1 + s_3^2 = 0$$

$$-y_2 + s_4^2 = 0$$

Construct the Lagrangian function,

$$L = L(y_1, y_2, s_1, s_2, s_3, s_4, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$$

$$= (2y_1 + 3y_2 - 2y_1^2) - \lambda_1(y_1 + 4y_2 + s_1^2 - 4) - \lambda_2(y_1 + y_2 + s_2^2 - 2)$$

$$-\lambda_3(-y_1 + s_3^2) - \lambda_4(-y_2 + s_4^2)$$

Since, $-2y_1^2$ represents negative semi definite quadratic form $Z = 2y_1 + 3y_2 - 2y_1^2$ is concave in y_1 and y_2. Thus maxima of L will be a maxima of $Z = 2y_1 + 3y_2 - 2y_1^2$ and vice-versa.

To derive the necessary and sufficient conditions for a maxima value of L (and hence of Z^*), we equate first order derivatives of L with respect to y_1, y_2, s_l and $\lambda_l, \ l = 1, 2, 3, 4.$ with zero. Thus we have,

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$$\frac{\partial L}{\partial y_1} = 2 - 4y_1 - \lambda_1 - \lambda_2 + \lambda_3 = 0$$

$$\frac{\partial L}{\partial y_2} = 3 - 4\lambda_1 - \lambda_2 + \lambda_4 = 0$$

$$\frac{\partial L}{\partial s_1} = -2\lambda_1 s_1 = 0, \qquad \frac{\partial L}{\partial \lambda_1} = y_1 + 4y_2 + s_1^2 - 4 = 0$$

$$\frac{\partial L}{\partial s_2} = -2\lambda_2 s_2 = 0, \qquad \frac{\partial L}{\partial \lambda_2} = y_1 + y_2 + s_2^2 - 2 = 0$$

$$\frac{\partial L}{\partial s_3} = -2\lambda_3 s_3 = 0, \qquad \frac{\partial L}{\partial \lambda_3} = -y_1 + s_3^2 = 0$$

$$\frac{\partial L}{\partial s_1} = -2\lambda_4 s_4 = 0, \qquad \frac{\partial L}{\partial \lambda_4} = -y_2 + s_4^2 = 0$$

After simplification, the above equations yields,

$$\left.\begin{aligned} &4y_1 + \lambda_1 + \lambda_2 - \lambda_3 = 2 \\ &4\lambda_1 + \lambda_2 - \lambda_4 = 3 \\ &y_1 + 4y_2 + s_1^2 = 4 \\ &y_1 + y_2 + s_2^2 = 2 \end{aligned}\right\| \;.....(1)$$

$$\left.\begin{aligned} &\lambda_1 s_1^2 + \lambda_2 s_2^2 + \lambda_3 y_1 + \lambda_4 y_2 = 0 \\ &y_1, y_2, s_1^2, s_2^2, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \end{aligned}\right\| \;....(2)$$

A solution $y_i,\ i = 1, 2$ to equation (1) satisfying (2) shall necessarily be an optimum one for maximizing.

To determine the solution to the above simultaneous equations (1), we introduce artificial variables $w_i,\ i = 1, 2$ (both non negative) in the respective constraints of (1) and construct the dummy objective function $Z = w_1 + w_2$

Thus the problem becomes

$$Maximise \quad g = w_1 + w_2$$

$$Subject \ to \ the \ contraints$$

$$\begin{aligned} &4y_1 + \lambda_1 + \lambda_2 - \lambda_3 + w_1 = 2 \\ &4\lambda_1 + \lambda_2 - \lambda_4 + w_2 = 3 \\ &y_1 + 4y_2 + s_1^* = 4 \\ &y_1 + y_2 + s_2^* = 2 \end{aligned}$$

$$y_1, y_2, s_1^*, s_2^*, w_1, w_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$$

Satisfying the complementary slackness condition

$$\lambda_1 s_1^* + \lambda_2 s_2^* + \lambda_3 y_1 + \lambda_4 y_2 = 0$$

Here, we have replaced s_1^2 by s_1^* and s_2^2 by s_2^*. The optimum solution to the above linear programming problem can be solved by using management scientist software, the optimum solution is:

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$$y_1 = 0.313,\ y_2 = 0.844,\ \ s_1^* = 0,\ \ s_2^* = 0,$$
$$\lambda_1 = 0.75, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0$$

Hence optimum solution to the original problem can be obtained as:

$$x_1 = \frac{1}{y_1} = 3.195,\ \ x_2 = \frac{1}{y_2} = 1.185\ \ and\ \ Z_{\min} = 2.962$$

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