ISSN: 2350-0328

International Journal of Advanced Research in Science, Engineering and Technology

Vol. 3, Issue 2 , February 2016

Finitely Quasi Injective and Quasi Finitely Injective S-systems Over Monoids

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ABSTRACT: The notion of quasiinjectivity relative to a class of finitely generated subsystems namely finitely quasi injective and quasi finitely injective systems over monoids are introduced and studied which are proper generalizations of quasi injective systems . Several properties of these kind of generalizations are discussed . Conditions under which subsystems of finitely quasi injective system inherit this property . Characterizations of finitely quasi injective and quasi finitely injective systems over monoids are considered . The relationship between the classes of finitely quasi injective with other classes of injectivity are studied. As a consequence, conditions to versus these classes are shown .

KEYWORDS: Finitely quasi-injective systems , Quasi finitely injective systems , Finitely injective systems , Finitely generated systems , Weakly injective systems.

I- INTRODUCTION AND PRELIMINARIES

Throughout this paper , the basic S-system is a unitary right S-system with zero which is consists of a monoid with zero , a non-empty set M_s with a function $f : M \times S \longrightarrow M$ such that $f(m,s) \longmapsto ms$ and the following properties hold : (1) $m \cdot 1 = m$ (2) $m(st) = (ms)t$ for all $m \in M$ and $s,t \in S$, where 1 is the identity element of S . An element $\Theta \in M_s$ is called fixed of M_s if $\Theta s=\Theta$ for all $s \in S$ [4]. An S-system M_s is centered if it has a fixed element Θ necessary unique such that $m0 = \Theta$ for all $m \in M_s$, where 0 is the zero element of S and Θ is the zero of M [8] . A subsystem N of an S-system M_s , is a non-empty subset of M such that $xs \in N$ for all $x \in N$ and $s \in S$[8]. Let g be a function from an S-system A_s into an S-system B_s , then g will be called an S-homomorphism , if for any $a \in A_s$ and $s \in S$, we have $g(as) = g(a)s$ [3]. An S-congruence ρ on a right S-system M_s is an equivalence relation on M_s such that whenever $(a,b) \in \rho$, then $(as, bs) \in \rho$ for all $s \in S$ [6]. The identity S-congruence on M_s will be denoted by I_M such that $(a,b) \in I_M$ if and only if $a = b$ [6] .

The authors defined that if for every $x \in M_s$, there is an S-homomorphism $f : M_s \rightarrow xS$ such that $x = f(x_1)$ for $x_1 \in M_s$, then an S-system M_s is called principal self-generator [1] . A subset A of an S-system M_s is called a set of generating elements or a generating set of M_s if every element $m \in M_s$ can be presented as $m = as$ for some $a \in A$, $s \in S$. Then , an S-system M_s is finitely generated if $M_s = < A >$ for some A , $|A| < \infty$, where $< A >$ is the subsystem of M_s generated by A[7 , p.63] . An S-system N_s is called M_s-generated , where M_s be an S-system if there exists an S-epimorphism $\alpha : M_s^{(I)} \longrightarrow N_s$ for some index set I . If I is finite , then N is called finitely M_s-generated of M_s [2] . An S-system B_s is a retract of an S-system A_s if and only if there exists a subsystem W of A_s and epimorphism $f : A_s \rightarrow W$ such that $B_s \cong W$ and $f(w) = w$ for every $w \in W$ [7, P.84] . An S-homomorphism f which maps an S-system M_s into an S-system N_s is said to be split if there exists S-homomorphism g which maps N_s into M_s such that $fg=1_N$ [6] .

Let A_s , M_s be two S-systems . A_s is called M_s-injective if given an S- monomorphism $\alpha : N \rightarrow M_s$ where N is a subsystem of M_s and every S-homomorphism $\beta : N \rightarrow A_s$,can be extended to an S-homomorphism $\sigma : M_s \rightarrow A_s$ [10] . An S-system A_s is injective if and only if it is M_s-injective for all S-systems M_s . An S-system A_s is quasi injective if and only if it is A_s-injective . Quasi injective S-systems have been studied by Lopez and Luedeman [8] . It is clear that every injective system is quasi injective but the converse is not true in general see [8] . An S-system A_s is weakly injective if it is injective relative to all embeddings of right ideals into S_s [7,p.205] .

ISSN: 2350-0328
International Journal of Advanced Research in Science, Engineering and Technology
Vol. 3, Issue 2 , February 2016

In this work, we find weak form of quasi injectivity called finitely quasi injective and quasi finitely injective systems over monoids . Also , we give some interesting results on these systems .

II-FINITELY QUASI INJECTIVE SYSTEMS OVER MONOIDS

In [9] , V.S.Ramamurthi define finitely injective module which motivate us to define finitely injective relative to S-system as follows :

Definition (2.1) : Let M_s and N_s be two S-systems . M_s is called finitely N_s-injective (for short F-N_s-injective) if every homomorphism from a finitely generated subsystem of N_s to M_s extends to homomorphism of N_s into M_s . An S-system M_s is called finitely quasi injective(for short FQ-injective) if M_s is F-M-injective system .

Example and Remarks(2.2) :

(1) Every quasi injective systems is FQ-injective systems , but the converse is not true in general as the following example shows ; let S be the monoid {1,a,b,0} with ab = a^2 = a and ba= b^2 =b . Now , consider S as a right S-system over itself , then it is easy to check that S_s is FQ- injective . But , when we take N={a,0} be a subsystem of S_s and f be S-homomorphism defined by $f(x) = \begin{Bmatrix} 0 & \text{if } x = 0 \\ b & \text{if } x = a \end{Bmatrix}$, then this S-homomorphism cannot be extended to S-endomorphism of S_s . If not , that is there exists S-homomorphism g: $S_s \to S_s$ such that g(x) = f(x) , for each $x \in N$, which is the trivial S-homomorphism , since other extension is not S-homomorphism . Then , b = f(a) = g(a) = a which implies that b = a , and this is a contradiction .

(2) Isomorphic system to F-M-injective is F-M-injective for any S-system M . In particular , isomorphic system to FQ-injective is FQ-injective .

(3) Let N_1 and N_2 be two S-systems such that $N_1 \cong N_2$. If M_s is F-N_1-injective , then M_s is F-N_2-injective .

In the following theorem , we give characterizations of FQ- injective S-systems :

Theorem (2.3) : The following statements are equivalent for S-system M_s with
$T = End_s(M_s)$:
(1) M_s is FQ-injective .
(2) $\gamma_{S_n}(x) \subseteq \gamma_{S_n}(y)$, where $x , y \in M^n$, $n \in Z^+$ implies that$Ty \subseteq Tx$.
(3) If $x_i \in M_s$, i = 1, 2, … , n and $\alpha : \dot{\cup}_{i=1}^{n} x_i S \to M_s$ is S-homomorphism , then there exists S-homomorphism extends α .

Proof: Put $M^n = M^{1\times n}$ and $S_n = S_{n\times 1}$.

(1→2) Let $\gamma_{S_n}(x) = \{ (s,s') \in S_n \mid xs = xs' \}$, where $s = \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}$ and $s' = \begin{pmatrix} s_1' \\ \vdots \\ s_n' \end{pmatrix}$} and $\gamma_{S_n}(x) = \gamma_{S_n}(y)$ such that $x = (x_1 , \dots, x_n)$, $y = (y_1 , \dots , y_n) \in M^n$, $n \in Z^+$. Then, $\alpha: \dot{\cup}_{i=1}^{n} x_i S \to M_s$is defined by α(xs) = ys . It is obvious that α is well-defined and S-homomorphism . Since M_s is FQ-injective , so there exists $\sigma \in T$ such that σ extends α , then $y_i = \alpha(x_i) = \sigma(x_i)$, where i = 1,2,…,n , so $y = \sigma x$ and then $Ty \subseteq Tx$.

(2→3) As α is S-homomorphism and β is S-monomorphism , so we have $\gamma_{s_n}(\beta(x_1), \dots, \beta(x_n)) \subseteq \gamma_{s_n}(\alpha(x_1), \dots, \alpha(x_n))$ by (2) , we have $T\alpha(x) \subseteq T\beta(x)$, where $\alpha(x) = (\alpha(x_1), \dots, \alpha(x_n)) = \alpha(x_1 , \dots , x_n)$ and $\beta(x) = (\beta(x_1), \dots, \beta(x_n)) = \beta(x_1 , \dots , x_n)$. Thus there exists $\sigma \in T$ such that $(\alpha(x_1), \dots, \alpha(x_n)) = \sigma(\beta(x_1), \dots, \beta(x_n))$, so$\alpha(x) = \sigma\beta(x)$. Therefore $\alpha = \sigma\beta$.

(3→1) By definition of FQ-injective system .

ISSN: 2350-0328
International Journal of Advanced Research in Science, Engineering and Technology
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Corollary (2.4) : The following statements are equivalent for a monoid S :

(1) S is a right F-injective .
(2) $\gamma_{S_n}(\alpha) \subseteq \gamma_{S_n}(\beta)$, where $\alpha , \beta \in S^n$, $n \in Z^+$ implies that $S\beta \subseteq S\alpha$.
(3) If $a_i \in S$, i = 1, 2, … , n and $\alpha : \dot{\cup}_{i=1}^{n} a_i S \to S$ is S-homomorphism , then there exists S-homomorphism b belong to S which is extends α .

The following proposition gives a condition under which subsystem of FQ-injective inherit this property . Before this , we need the following concept :

Recall that a subsystem N of S-system M_s is fully invariant of M_s if $f(N) \subseteq N$, for all $f \in End_s(M_s)$ [5] . An S-system is called duo if each subsystem of it is fully invariant .

Proposition (2.5) : Every fully invariant subsystem ofFQ-injective system is FQ-injective .

Proof: Let M_s be FQ-injective system and N be a fully invariant subsystem of M_s . Let X be any finitely generated subsystem of N and f be S-homomorphism from X into N . Since M_s is FQ-injective system , so there exists an S-endomorphism g of M_s such that $goi_N oi_X = i_N of$, where i_X and i_N are the inclusion maps of X into N and N into M_s respectively . As N is fully invariant in M_s , so $g(N) \subseteq N$. Put $g_{|N} = h$, then$\forall$ $x \in X$, we have $(hoi_X)(x) = g(x) = (goi_N oi_X)(x) = (i_N of)(x) = f(x)$. Therefore N is FQ-injective system .

Recall that an S-system M_s is called multiplication if every subsystem of M_s is of the form MI for some right ideal I of S . It is clear that every subsystem of multiplication system is fully invariant [5] .

Corollary (2.6) : If M_s is FQ-injective duo (multiplication) S-system , then every subsystem of M_s is FQ-injective .

Proposition (2.7) : Let M_s and N_s be two S-systems and $N^/$ a subsystem of N_s . If M_s is F-N-injective, then :

(1) Every retract of M_s is F-N-injective.
(2) M_s is F-$N^/$-injective system .
Proof :(1) Let $M_s = M_1 \oplus M_2$, and K be finitely generated subsystem of N and f be S-homomorphism of K into M_1 . Since M_s is F-N_s-injective, so ($j_1$of) where j_1 is injection of M_1 into M_s extends to S-homomorphism g of N_s into M_s such that $goi_K = j_1 of$. Put $g^/ (= \pi_1 g) : N_s \to M_1$, where π_1 be the projection map of M_s into M_1 , then $g^/ oi_K = \pi_1 ogoi_K = \pi_1 oj_1 of = I_{M_1} of = f$. Thus f extends to S-homomorphism $g^/$ and M_1 is F-N-injective system.

(2) It is obvious .

The following corollaries is immediately from above proposition :
Corollary (2.8): Retract of FQ-injective system is FQ-injective .

Corollary (2.9) : Let N be any subsystem of S-system M_s . If N is F-M-injective , then N is finitely injective .

Proposition (2.10) : Let M_s and N_s be two S-systems . Let N_s be finitely generated subsystem of M_s . Then N_s is F-M-injective if and only if every monomorphismf : $N_s \to M_s$ split .

Proof: Assume that N_s is F-M_s-injective system and f : $N_s \to M_s$ be monomorphism , then by F-M_s-injective of N_s , there exists an S-homomorphism g : $M_s \to N_s$ such that $gof = I_N$. Since $N_s \cong f(N_s)$, so $f(N_s)$ is a retract of M_s . Conversely , assume that A is finitely generated subsystem of M_s . Then , by assumption the monomorphism (inclusion map) i_A of A into M_s split , this means there exists $\omega : M_s \to A$ such that $\omega oi_A = I_A$. Now , for S-homomorphism f : A $\to N_s$, set $g (= fo\omega) : M_s \to N_s$ which implies that $goi_A = f o\omega oi_A = f oI_A = f$. Thus N_s is F-M-injective system .

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International Journal of Advanced Research in Science, Engineering and Technology
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Corollary (2.11) : Let N_s be a finitely generated subsystem of an S-system M_s . If N_s is F-M_s-injective system , then N_s is a retract of M_s .

Corollary (2.12) : Let M_s be FQ-injective S-system . Then , every finitely generated subsystem of M_s which is isomorphic to M_s is a retract of M_s .

Definition (2.13) : An S-system M_s is called FC_2 if every finitely generated subsystem of M_s that is isomorphic to a retract of $M_s$is itself a retract ofM_s .

Theorem (2.14) : Every FQ-injective system satisfies FC_2 .

Proof: Let M_s be FQ-injective S-system and A be a retract of M_s with $A \cong B$, where B is finitely generated subsystem of M_s . Let f be S-isomorphism from B into A, then f is S-monomorphism from B into M_s . Since A is a retract of M_s , so by corollary(2.8) A is F-M-injective system . By example and remarks (2.2)(2) , since $A \cong B$, so B is F-M-injective system . Then , by proposition (2.10) f is split and by corollary (2.9) B is a retract of M_s and so M_s satisfies FC_2 – condition .

Proposition(2.15) : Let M_s be an S-system and $\{N_i\}_{i\in I}$ be a family of S-systems , where I is finite index set . Then ,$\Pi_{i\in I}N_i$ is finitely M-injective if and only if for each $i \in I$, N_i is finitely M-injective system .

Proof:⇒) Put $N_s = \Pi_{i\in I}N_i$, assume that N_s is F-M-injective S-system and A is a finitely generated subsystem of M_s . Let f be an S-homomorphism of A into N_i. Since N is F-M-injective , so there exists S-homomorphism $g : M_s \rightarrow N_s$ such that $goi_A = j_iof$, where j_i is the injection map of N_i into N_s andi_A is the inclusion map of A into M_s . Now , let π_i be the projection map of N onto N_i. Put
$h(=\pi_i og): M_s\rightarrow N_i$, then $\forall a \in A$, $(hoi_A)(a) = (\pi_i ogoi_A)(a) = (\pi_i oj_iof)(a) = f(a)$. Thus N_i is F-M-injective system .

⇐) Assume that N_i is F-M-injective for each $i \in I$. Let A be finitely generated subsystem of M_s and f be an S-homomorphism of A into N_s . Since N_i is F-M-injective S-system , so there exists S-homomorphism $\beta_i : M_s \rightarrow N_i$ such that $\beta_i oi_A = \pi_i of$, where i_A be the inclusion map of A into M_s . Now, define an S-homomorphism $\beta (= j_io\beta_i) : M_s \rightarrow N_s$, then $\beta oi_A = j_io\beta_i oi_A = j_io\pi_i of = f$. Therefore , N_s is F-M-injective system .

Corollary (2.16) : Let M_s and N_i be S-systems , where $i \in I$ and I is finite index set . If $\oplus_{i\in I}N_i$ is F-M-injective for all $i \in I$, then N_i is F-M-injective .

The following proposition give another characterization of FQ-injective S-system :

Proposition (2.17): If M_s isFQ-injective S-system and $T = End(M_s)$, then TA = TB for each isomorphic subsystems A and B of M_s .

Proof : By assumption there exists an S-isomorphism $\alpha : A \rightarrow B$, let $b \in B$ so there exists $a \in A$ such that $\alpha(a) = b$. For $s,t\in S$, if as = at and since α is well-defined , so $\alpha(as) = \alpha(at)$, then bs = bt , which implies that $\gamma_s(a) \subseteq \gamma_s(b)$. Since M_s is FQ-injective, then by theorem (2.3) , $Tb \subseteq Ta$ and hence $Tb \subseteq TA$ $\forall b \in B$. Thus $TB \subseteq TA$. Similarly , we can prove $TA \subseteq TB$. Therefore TA = TB .

As an immediate consequence of above proposition , we have the following result :

Corollary (2.18): If S is F-injective monoid and A , B are two isomorphic ideal of S , then A = B .

Recall that two S-systems M_s and N_s are mutually finitely injective if M_s is finitely N_s-injective and N_s is finitely M-injective .

Theorem (2.19) : If $M_1\oplus M_2$ is FQ-injective system , then M_1 and M_2 are mutually F-injective system .

ISSN: 2350-0328

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Proof : Let $M_1\oplus M_2$ be FQ-injective system . Let X be any finitely generated subsystem of M_2 and f be S-homomorphism from X into M_1 . Put $\alpha(=j_i of):X\rightarrow M_1\oplus M_2$, where j_i is the injection map of M_1 into $M_1\oplus M_2$. By proposition (2.7)(2) , $M_1\oplus M_2$ is F-M_2-injective , so α extends to S-homomorphism g : $M_2 \rightarrow M_1\oplus M_2$. If π_1 : $M_1\oplus M_2\rightarrow M_1$ is the natural projection , then $h(=\pi_1 g) : M_2 \rightarrow M_1$ is S-homomorphism extending f . Consequently , M_1 is F-M_2-injective system .

The proof of the following corollary is immediately from above theorem and proposition (2.7) :

Corollary (2.20) : If $\oplus_{i\in I}M_i$ is FQ-injective system , then M_j is F-M_K-injective for all distinct j , k ∈ I .

Definition (2.21) : An S-system M_s is called quasi finitely injective (for short QF-injective) if every S-homomorphism from a finitely M_s-generated subsystem of M_s to M_s extends to an S-endomorphism of M_s .

Proposition (2.22) : The following statements are equivalent for S-system M_s with
$T = End_s(M_s)$:
(1) M_s is QF-injective .
(2) $\gamma_{M_n}(\alpha) \subseteq \gamma_{M_n}(\beta)$, where α , β ∈ T^n , n ∈ Z^+ implies that $T\beta\subseteq T\alpha$.

Proof : (1→2) Assume that $\gamma_{M_n}(\alpha) \subseteq \gamma_{M_n}(\beta)$ such that α,β ∈ T^n , n ∈ Z^+. Write $\alpha = (\alpha_1 , \dots , \alpha_n)$, $\beta = (\beta_1 , \dots , \beta_n)$, then the mapping $f : \dot{\bigcup}_{i=1}^{n}\alpha_i M \rightarrow M_s$ defined by $f(\alpha_i m) = \beta_i m$ is well-defined and S-homomorphism , for this let $\alpha_i m = \alpha_i k \forall\, i \in I$, so $(m, k) \in \gamma_{M_n}(\alpha) \subseteq \gamma_{M_n}(\beta)$ which implies that $\beta_i m = \beta_i k$ and then $f(\alpha_i m) = f(\alpha_i k)$. Also , for S-homomorphism ,we have $f(\alpha_i m)\, s = \beta_i ms = f(\alpha_i ms)$. Since M_s is QF-injective , so there exists S-endomorphism g of M_s which extends f , then $\beta_i m = g(\alpha_i m) = f(\alpha_i m)$,∀ i ∈ I and $m \in M_s$. Thus β = gα and so Tβ ⊆ Tα .
(2→1) Assume that $f : \dot{\bigcup}_{i=1}^{n}\alpha_i M \rightarrow M_s$ be homomorphism . Put $\alpha = (\alpha_1 , \dots , \alpha_n)$, $\beta = (f\beta_1 , \dots , f\beta_n)$, then it is easy matter to check that $\gamma_{M_n}(\alpha) \subseteq \gamma_{M_n}(\beta)$. By(2) , we have β ∈ Tα , so there exists σ ∈ T such that $\beta = \sigma\alpha$. Since f(α(M)) = β(M) = σα(M) . Thus σ extends f .

The following proposition give a condition under which endomorphism of S-system is QF-injective :

Proposition (2.23) : Given an S-system M_s with $T = End_s(M_s)$. Let α , β denote elements of T . Assume that $M_s\times M_s$ generates kerα for each α ∈ T . Then T is right QF-injective if and only if kerα ⊆ kerβ implies that β ∈ Tα .

Proof : If T is right QF-injective , then the condition holds for any M_s . Conversely , if β ∈ ℓ_T(kerα) = Tα , so there exists σ ∈ T such that β = σα . The proof is complete when we prove kerα ⊆ kerβ . Since $M_s\times M_s$ generates kerα , so there exists S-epimorphism $f_i : M_s\times M_s \rightarrow$ kerα such that ∀(x,y) ∈ kerα , we have α(x) = α(y) , and then there exists (m, k) ∈ $M_s\times M_s$, where $x = f_i m$, $y = f_i k$. Now , since σ is well-defined , so σα(x) = σα(y) which implies that β(x) = β(y) and (x,y) ∈ kerβ . Thus T is QF-injective by proposition (2.22) .

The following proposition give a condition under which endomorphism of QF-injective system is F-injective :

Proposition (2.24) : Let M_s be a right S-system with $T = End_s(M_s)$, then :
(1) If T is right F-injective , then M_s is QF-injective .
(2) If M_s is QF-injective and $M_s\times M_s$ generates $\gamma_{M_n}(\alpha)$ for any positive integer n and α ∈ T^n , then T is right F-injective .

Proof :(1) Let $\gamma_{M_n}(\alpha) \subseteq \gamma_{M_n}(\beta)$, where α , β ∈ T^n , n ∈ Z^+ , then $\gamma_{T_n}(\alpha) \subseteq \gamma_{T_n}(\beta)$. Since T is right F-injective , so by corollary (2.4) we have Tβ⊆ Tα . Then , by proposition (2.26) M_s is QF-injective system .

(2) Let $\gamma_{T_n}(\alpha) \subseteq \gamma_{T_n}(\beta)$, where α , β ∈ T^n , n ∈ Z^+ . Then , for any (x,y) ∈ $\gamma_{M_n}(\alpha)$, we have α(x) = α(y) . Since $M_s\times M_s$ generates $\gamma_{M_n}(\alpha)$, so $x = \lambda_i m$, $y = \lambda_i k$,where (m , k) ∈ $M_s\times M_s$ and $\lambda_i \in T_n$. Then , $(\lambda_i m , \lambda_i k) \in \gamma_{T_n}(\alpha) \subseteq \gamma_{T_n}(\beta)$, so

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$\beta(\lambda_i m) = \beta(\lambda_i k)$. This means that $\beta(x) = \beta(y)$ and $(x,y) \in \gamma_{M_n}(\beta)$. Hence $\gamma_{M_n}(\alpha) \subseteq \gamma_{M_n}(\beta)$. Since M_s is QF-injective system , so $T\beta \subseteq T\alpha$ and consequently , T is F-injective by corollary (2.4) .

III -RELATIONSHIP AMOG FQ-INJECTIVE AND QF-INJECTIVE S-SYSTEMS WITH OTHER CLASSES OF INJECTIVITY

The following proposition gives a condition under which FQ-injective system is QF-injective system , but before this we need the following concept :

Definition (3.1) : An S-system M_s is called self-generator if it generates all its subsystems .

Proposition (3.2) : If M_s is finitely generated S-system which is self-generator , then M_s is FQ-injective system if and only if M_s QF-injective .

Proof: Assume that M_s is FQ-injective system . Let X be finitely M_s-generated subsystem of M_s and f be S-homomorphism of X into M_s . Since M_s is finitely generated and X is finitely M_s-generated , so there exists S-epimorphism $\alpha : M_s \to X$, so X is finitely generated . Since M_s is FQ-injective system , so f extends to S-endomorphism g of M_s such that $goi_X = f$, where $i_X$is the inclusion map of X into M_s and then M_s is QF-injective system . Conversely , assume that Assume that M_s is QF-injective system . Let A be finitely generated subsystem of M_s and f be S-homomorphism of A into M_s . Since M_s is self-generator , so there exists S-epimorphism $\alpha:M_s\to A$, and then A is finitely M_s-generated . Since M_s is QF-injective system , so f extends to S-endomorphism g of M_s such that $goi_A = f$, where $i_A$is the inclusion map of A into M_s and then M_s is FQ-injective system .

The following proposition explain under which condition on finitely $E(M_s)$-injective to be injective :

Proposition (3.3) : Let M_s be a finitely generated S-system . Then M_s is injective system if and only if M_s is finitely $E(M_s)$-injective .

Proof: $\Rightarrow$) It is obvious .
$\Leftarrow$) Let M_s be finitely $E(M_s)$-injective and f be S-monomorphism from M_s into $E(M_s)$. Since M_s is finitely $E(M_s)$-injective , so by proposition(2.10) , there exists an S-homomorphism $g:E(M_s)\to M_s$ such that $gof = I_M$ which implies that f is split and $f(M_s)$ is retract of $E(M_s)$, as $f(M_s) \cong M_s$. This implies that M_s is a retract of $E(M_s)$ and since $E(M_s)$ is injective , so M_s is injective .

As a particular case of above proposition , we have the following corollary :

Corollary (3.4) : A monoid S is self-injective if and only if S is finitely S-injective S-system .

The following proposition explain under which condition on FQ- injective to being injective , but before this we need the following concept :

Definition(3.6) : An S-system M_s is said to be weakly injective if for every finitely generated subsystem N of $E(M_s)$, we have $N \subseteq X \subseteq E(M_s)$ for some $X \cong M_s$.

Proposition (3.7) : Let M_s be a finitely generated system . Then M_s is injective system if and only if M_s is weakly injective and FQ-injective .
Proof: $\Longrightarrow$)It is obvious .
$\Longleftarrow$) It is enough to prove that $M_s = E(M_s)$. Let $x \in E(M_s)$, so $M_s \cup xS$ is finitely generated . As M_s is weakly injective , so there exists subsystem X of $E(M_s)$ such that $M_s \cup xS \subseteq X \cong M_s$. Since M_s is FQ-injective system , so X is also FQ-injective by example and remarks (2.2)(2) . By theorem(2.14) X is satisfy FC_2 and since M_s is finitely generated subsystem of X , so M_s is a retract of X . But M_s is $\cap$-large subsystem of $E(M_s)$, soM_s is $\cap$-large in X . Therefore M_s = X , and $x \in M_s$. Thus , $M_s = E(M_s)$ is injective .

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