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Subclasses of Bi-Univalent Functions Associated with Generalized Hypergeometric Function

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ABSTRACT: In this paper, we have introduced and investigated two new subclasses of the function class Δ of bi-univalent functions defined in the open unit disk, which are associated with the generalized Hypergeometric function. Furthermore, we find estimates on the Taylor-Maclaurin coefficient $| a_2 |$ and $| a_3 |$ for the functions belonging to these new classes.

KEYWORDS: Bi-univalent function, Hypergeometric function, Taylor-Maclaurin coefficient

I. INTRODUCTION

Let C(k) denote the class of the functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n , \tag{1}$$

which are analytic in the open unit disc $U = \{z: |z| < 1\}$. Further, by S we shall denote the class of all functions in C(k) which are univalent in U. Let f ∈ C (k) given by (1) and g ∈ C (k) given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad .$$

We define the convolution product (or Hadamard) of f and g by

$$(f*g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g*f)(z); (z \in U). \tag{2}$$

Some of the important and well-investigated subclasses of the univalent function class S include the class $S^*(\beta)$ of starlike functions of order β in U and the class K(β) of convex functions of order β in U which are defined as

$$S^*(\beta) = \left\{ f \in S: Re\left(\frac{zf'(z)}{f(z)}\right) > \beta \ \ (0 \le \beta < 1 \ ; z \in \mathcal{U}) \right\} \tag{3}$$

and

$$K(\beta) = \left\{ f \in C(k): Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta \ (0 \le \beta < 1 ; z \in \mathcal{U}) \right\} \tag{4}$$

It readily follows from the definition (3) and (4) that

$$f \in K(\beta) \ \Leftrightarrow zf' \in S^*(\beta).$$

It is well known that every function $f \in S$ have inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z, \qquad z \in U$$

And

$$f(f^{-1}(w)) = w , |w| < r_0(f) \ge 1/4,$$

Where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_3^3 - 5a_2 a_3 + a_4)w^4 + \ldots .... \tag{5}$$

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A function $f \in C(k)$ is said to be bi-univalent in U if both f(z) and $f^{-1}(z)$ are univalent in U. Let A denote the class of bi-univalent functions in U given by (1). For the complex parameters a, b and c with $c \neq 0, -1, -2, \dots \dots \dots$ the generalized Hypergeometric function $_2R_1(a, b, c, k; z)$ is defined as

$$_2R_1(a,b,c,k;z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n\, \Gamma(b+kn) z^n}{\Gamma(c+kn)\,(n)!} = 1 + \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_{n-1}\, \Gamma(b+k(n-1)) z^{n-1}}{\Gamma(c+k(n-1))\,(n-1)!} \qquad (6)$$

Where Re (c- 1-b) >0, |z| < 1 and $(a)_n$ is the Pochhammer symbol. By using generalized Hypergeometric function given by (6) we define a convolution operator $\Theta(a, b, c: k)$ as follows:

$$\Theta(a,b,c:k) f(z) = z \quad {}_2R_1(a,b,c,k;z) * f(z) = z + \sum_{n=2}^{\infty} \Upsilon_n a_n z^n \qquad (z \in U) \qquad (7)$$

Where

$$\Upsilon_n = \frac{\Gamma(c)(a)_{n-1}\, \Gamma(b+k(n-1))}{\Gamma(b)\, \Gamma(c+k(n-1))(n-1)!} \qquad (8)$$

Definition 1:- A function f(z) defined by (1) is said to be in the class M_A (a, b, c, k; α, λ) if the following condition are satisfied:

$$\left| arg\left(\frac{z\left(\Theta\,(a,b,c;k) f(z)\right)'}{(1-\lambda)z + \lambda\, \Theta\,(a,b,c;k) f(z)} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1; 0 \leq \lambda \leq 1;\ z \in U) \qquad (9)$$

And

$$\left| arg\left(\frac{w\left(\Theta\,(a,b,c;k) g(w)\right)'}{(1-\lambda)w + \lambda\, \Theta\,(a,b,c;k) g(w)} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1; 0 \leq \lambda \leq 1;\ w \in U) \qquad (10)$$

Where the function g is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_3^3 - 5a_2 a_3 + a_4) w^4 + \dots \dots \qquad (11)$$

That is, the extension of f^{-1} to U.

Definition 2:- A function f(z) defined by (1) is said to be in the class N_A (a, b, c, k; β, λ) if the following condition are satisfied:

$$Re\left(\frac{z\left(\Theta\,(a,b,c;k) f(z)\right)'}{(1-\lambda)z + \lambda\, \Theta\,(a,b,c;k) f(z)} \right) > \beta \;(0 \leq \beta < 1; 0 \leq \lambda \leq 1;\ z \in U) \qquad (12)$$

And

$$Re\left(\frac{w\left(\Theta\,(a,b,c;k) g(w)\right)'}{(1-\lambda)w + \lambda\, \Theta\,(a,b,c;k) g(w)} \right) > \beta \;(0 \leq \beta < 1; 0 \leq \lambda \leq 1;\ w \in U) \qquad (13)$$

Where the function g is given by (11)

In order to prove our main results, we shall need the following lemma

Lemma 1:- [2] if $h \in P$, then $|c_k| \leq 2$ for each k, where P is the family of all functions h, analytic in U, for which

$$Re(h(z)) > 0 \quad (z \in U),$$

where

$$h(z) = 1 + c_1 z + c_2 z^2 + \dots \dots \qquad (z \in U)$$

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II Coefficient Estimate for the Function class M_A (a, b, c, k; α, λ)

Theorem 1:- Let the function f(z) defined by (1) be in the class M_A (a, b, c, k; α, λ) for $0 < \alpha \leq 1; 0 \leq \lambda \leq 1$, then

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$$|a_2| \leq \frac{2\,\alpha}{\sqrt{[2\alpha(\lambda^2-2\lambda)+(1-\alpha)(2-\lambda)^2]\,\Upsilon_2^2+2\alpha\,(3-\lambda)\Upsilon_3}} \tag{14}$$

and

$$|a_3| \leq \frac{2\,\alpha}{(3-\lambda)\Upsilon_3} \tag{15}$$

Proof: it follows from (9) and (10) that

$$\frac{z\,(\Theta\,(a,b,c;k)f(z))'}{(1-\lambda)z+\lambda\,\Theta\,(a,b,c;k)f(z)} = [p(z)]^{\alpha} \tag{16}$$

And

$$\frac{w\,(\Theta\,(a,b,c;k)g(w))'}{(1-\lambda)w+\lambda\,\Theta\,(a,b,c;k)g(w)} = [q(w)]^{\alpha} \tag{17}$$

Where p(z) and q(w) have the following forms:

$$p(z) = 1 + p_1 z + p_2 z^2 + \ldots\ldots\ldots \tag{18}$$

And

$$q(w) = 1 + q_1 w + q_2 w^2 + \ldots\ldots\ldots \tag{19}$$

Respectively. Now, equating the coefficient in (16) and (17), we get

$$(2-\lambda)\Upsilon_2\,a_2 = \alpha\,p_1 \tag{20}$$

$$(\lambda^2-2\lambda)\Upsilon_2^2\,a_2^2 + (3-\lambda)\Upsilon_3\,a_3 = \frac{1}{2}[\alpha(\alpha-1)p_1^2 + 2\alpha\,p_2] \tag{21}$$

$$-(2-\lambda)\Upsilon_2 a_2 = \alpha\,q_1 \tag{22}$$

And

$$(\lambda^2-2\lambda)\Upsilon_2^2\,a_2^2 + (3-\lambda)\Upsilon_3\,(2a_2^2 - a_3) = \frac{1}{2}[\alpha(\alpha-1)q_1^2 + 2\alpha\,q_2] \tag{23}$$

From (20) and (22), we find that

$$a_2 = \frac{\alpha\,p_1}{(2-\lambda)\,\Upsilon_2} = \frac{-\alpha\,q_1}{(2-\lambda)\Upsilon_2} \tag{24}$$

Which implies

$$p_1 = -q_1 \tag{25}$$

Adding (21) and (23), we obtain

$$[2(\lambda^2-2\lambda)\Upsilon_2^2 + 2(3-\lambda)\Upsilon_3]\,a_2^2 = \frac{\alpha(\alpha-1)}{2}\,(p_1^2+q_1^2) + \alpha\,(p_2+q_2) \tag{26}$$

Substituting the values from (24) and (26) into (26), we get

$$p_1^2 = \frac{(2-\lambda)^2\,\Gamma_2^2\,(p_2+q_2)}{[2\alpha(\lambda^2-2\lambda)+(1-\alpha)(2-\lambda)^2]\,\Upsilon_2^2+2\alpha\,(3-\lambda)\Upsilon_3} \tag{27}$$

Applying Lemma 1 for the coefficient p_2 and q_2, we immediately have

$$|p_1| \leq \frac{2\alpha}{\sqrt{[2\alpha(\lambda^2-2\lambda)+(1-\alpha)(2-\lambda)^2]\,\Upsilon_2^2+2\alpha\,(3-\lambda)\Upsilon_3}} \tag{28}$$

Substituting (28) in (24), we get

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$$|a_2| \leq \frac{2\,\alpha}{\sqrt{[2\alpha(\lambda^2-2\lambda)+(1-\alpha)(2-\lambda)^2]\,\Upsilon_2^2+2\alpha\,(3-\lambda)\Upsilon_3}}$$

This gives the bound on $|a_2|$ as asserted in (14). Next, in order to find that bound on $|a_3|$, by subtracting (23) fro (21), we get

$$2(3-\lambda)\Upsilon_3 a_3 - 2(3-\lambda)\Upsilon_3 a_2^2 = \alpha\,(p_2 - q_2) + \frac{\alpha(\alpha-1)}{2}\,(p_1^2 - q_1^2) \qquad (29)$$

It follows from (24), (25) and (29) that

$$2(3-\lambda)\Upsilon_3 a_3 = \left[\frac{2(3-\lambda)\alpha^2\,\Upsilon_3}{[2\alpha(\lambda^2-2\lambda)+(1-\alpha)(2-\lambda)^2\,]\,\Upsilon_2^2+2\alpha\,(3-\lambda)\Upsilon_3}+\alpha\right] p_2 + \left[\frac{2(3-\lambda)\alpha^2\,\Upsilon_3}{[2\alpha(\lambda^2-2\lambda)+(1-\alpha)(2-\lambda)^2\,]\,\Upsilon_2^2+2\alpha\,(3-\lambda)\Upsilon_3}-\alpha\right] q_2$$

Applying lemma 1 once again for the coefficient p_2 and q_2 , we readily get

$$|a_3| \leq \frac{2\,\alpha}{(3-\lambda)\Upsilon_3}$$

This completes the proof of the theorem.
□

Putting $\lambda = 0$ in theorem 1, we have the following Corollary.

Corollary 1:- Let the function f (z) defined by (1) be in the class M_A (a, b, c, k; α) ($0 < \alpha \leq 1$), then

$$|a_2| \leq \alpha\,\sqrt{\frac{2}{2(1-\alpha)\Upsilon_2^2+3\alpha\,\Upsilon_3}}$$

And

$$|a_3| \leq \frac{2\,\alpha}{3\,\Upsilon_3}$$

Putting $\lambda = 0$, a=c and b =1 in Theorem 1, we have the following Corollary

Corollary 2:- Let the function f (z) defined by (1) be in the class M_A (a, k; α) ($0 < \alpha \leq 1$), then

$$|a_2| \leq \alpha\,\sqrt{\frac{2}{2+\alpha}}$$

And

$$|a_3| \leq \frac{2\,\alpha}{3}$$

The bound on $|a_3|$ in Corollary 2 provides improvement over the result of Srivastava et. Al. [3].

Putting $\lambda = 1$ in Theorem 1, we have the following Corollary

Corollary 3:- Let the function f (z) defined by (1) be in the class M_A (a, b, c, k; α, 1) ($0 < \alpha \leq 1$), then

$$|a_2| \leq \alpha\,\sqrt{\frac{2}{(1-3\alpha)\Upsilon_2^2+4\alpha\,\Upsilon_3}}$$

And

$$|a_3| \leq \frac{\alpha}{\Upsilon_3}$$

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III Coefficient Estimate for the Function class N_A (a, b, c, k; β, λ)

Theorem 2:- Let the function f(z) defined by (1) be in the class N_A (a, b, c, k; β, λ) for $0 \le \beta < 1; 0 \le \lambda \le 1$, then

$$|a_2| \le \sqrt{\frac{2(1-\beta)}{(\lambda^2-2\lambda)\Upsilon_2^2+(3-\lambda)\Upsilon_3}} \qquad (30)$$

and

$$|a_3| \le \frac{2\,(1-\beta)}{(3-\lambda)\Upsilon_3} \qquad (31)$$

Proof: it follows from (12) and (13) that

$$\frac{z\left(\Theta\,(a,b,c;k)f(z)\right)'}{(1-\lambda)z+\lambda\,\Theta\,(a,b,c;k)f(z)} = \beta+(1-\beta)p(z) \qquad (32)$$

And

$$\frac{w\left(\Theta\,(a,b,c;k)g(w)\right)'}{(1-\lambda)w+\lambda\,\Theta\,(a,b,c;k)g(w)} = \beta+(1-\beta)q(w) \qquad (33)$$

Where p(z) and q(w) have the forms (18) and (19) respectively. Equating the coefficient in (32) and (33), we get

$$(2-\lambda)\Upsilon_2\,a_2 = (1-\beta)\,p_1 \qquad (34)$$
$$(\lambda^2-2\lambda)\Upsilon_2^2\,a_2^2+(3-\lambda)\Upsilon_3\,a_3 = (1-\beta)\,p_2 \qquad (35)$$
$$-(2-\lambda)\Upsilon_2 a_2 = (1-\beta)\,q_1 \qquad (36)$$

And

$$(\lambda^2-2\lambda)\Upsilon_2^2\,a_2^2+(3-\lambda)\Upsilon_3\,(2a_2^2-a_3) = (1-\beta)\,q_2 \qquad (37)$$

From (34) and (36), we find that

$$a_2 = \frac{(1-\beta)\,p_1}{(2-\lambda)\,\Upsilon_2} = \frac{-(1-\beta)\,q_1}{(2-\lambda)\Upsilon_2} \qquad (38)$$

Which implies

$$p_1 = -q_1 \qquad (39)$$

From (35) and (37), we obtain

$$[2(\lambda^2-2\lambda)\Upsilon_2^2+2(3-\lambda)\Upsilon_3]\,a_2^2 = (1-\beta)\,(p_2+q_2) \qquad (40)$$

Also by using (38) and (40), we get

$$p_1^2 = \frac{(2-\lambda)^2\,\Gamma_2^2\,(p_2+q_2)}{[(\lambda^2-2\lambda)^2\,\Upsilon_2^2+2\alpha\,(3-\lambda)\Upsilon_3](1-\beta)} \qquad (41)$$

Applying Lemma 1 in (41) appropriately, we get

$$|p_1| \le (2-\lambda)\,\Upsilon_2\sqrt{\frac{2}{[(\lambda^2-2\lambda)^2\,\Upsilon_2^2+2\alpha\,(3-\lambda)\Upsilon_3](1-\beta)}} \qquad (42)$$

Again by applying lemma 1 to (38) and using (42), we immediately find that

$$|a_2| \le \sqrt{\frac{2(1-\beta)}{(\lambda^2-2\lambda)^2\,\Upsilon_2^2+2\alpha\,(3-\lambda)\Upsilon_3}}$$

This gives the bound on $|a_2|$ as asserted in (30). Next, in order to find that bound on $|a_3|$, by subtracting (37) fro (35), we get

$$2(3-\lambda)\Upsilon_3 a_3 - 2(3-\lambda)\Upsilon_3 a_2^2 = (1-\beta)(p_2 - q_2) \quad (43)$$

It follows from (40) and (43) that

$$2(3-\lambda)\Upsilon_3 a_3 = \left[\frac{2(3-\lambda)\,\Upsilon_3\,(1-\beta)}{(\lambda^2-2\lambda)\quad \Upsilon_2^2+(3-\lambda)\Upsilon_3}\right] p_2 + \left[\frac{(\lambda^2-2\lambda)\quad \Upsilon_2^2\,(1-\beta)}{(\lambda^2-2\lambda)\quad \Upsilon_2^2+(3-\lambda)\Upsilon_3}\right] q_2$$

Applying lemma 1 once again for the coefficient p_2 and q_2 , we readily get

$$|a_3| \leq \frac{2\,(1-\beta)}{(3-\lambda)\Upsilon_3}$$

This completes the proof of the theorem.

□

Putting $\lambda = 0$ in theorem 2, we have the following Corollary.

Corollary 4:- Let the function f (z) defined by (1) be in the class N_A (a, b, c, k; β) $(0 \leq \beta < 1)$, then

$$|a_2| \leq \alpha\sqrt{\frac{2\,(1-\beta)}{3\,\Upsilon_3}}$$

And

$$|a_3| \leq \frac{2\,(1-\beta)}{3\,\Upsilon_3}$$

Putting $\lambda = 0$, a=c and b =1 in Theorem 2, we have the following Corollary

Corollary 5:- Let the function f (z) defined by (1) be in the class N_A (a, k; β) $(0 \leq \beta < 1)$, then

$$|a_2| \leq \alpha\sqrt{\frac{2\,(1-\beta)}{2\,\Upsilon_3-\Upsilon_2^2}}$$

And

$$|a_3| \leq \frac{(1-\beta)}{\Upsilon_3}$$

The bound on $|a_3|$ in Corollary 5 provides improvement over the result of Srivastava et. Al. [3].

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