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Numerical Solution of Stochastic Differential Equations

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ABSTRACT: This paper provides an introduction to the main concepts and techniques necessary for someone who wishes to carryout numerical experiments involving Stochastic Differential Equation (SDEs). As SDEs are frictionless generally and the solutions are continuous stochastic process that represent diffusive dynamic especially in finance, it is required of us to take into account random effects and influences in real world systems which are essential in the accurate description of such situations.

We include a review of Stochastic Differential equations (SDE), Geometric Brownian Motion, Euler-Maruyama, Milstein and Taylor approximate which gives a clear picture of their graphical approximate and exact solution. We finally compared the convergence of Euler-Maruyama and Milstein

KEYWORDS: Stochastic Differential Equations, Stochastic Taylor Expansion, Euler-Maruyama, path wise Approximation and Strong convergence, Weak Euler Scheme and Milstein Method

I. INTRODUCTION

Stochastic Differential Equations(SDEs) are differential equations where stochastic process represents one or more terms and, as a result consequence; the resultant solution will also be stochastic [3].As more realistic, Mathematical Models become required to take into account random effects and influences in real world systems and SDEs have become essential in the accurate description of such situations [2]. The solutions are continuous-time stochastic processes and methods for the computational solution of stochastic differential equation are based on similar techniques for stochastic dynamic [3].

Stochastic modelling has come to play an important role in many branches of science and industry. The concept has been initiated by Einstein in 1905 [12]. In his article, he presented a mathematical connection between microscopic random motion of a particles and macroscopic diffusion equation. The models have been used after with great success in a variety of application areas, including biology, epidemiology, mechanics, economics and finance. Various authors have given their contribution in these field.Kloeden and Platen[8] have discussed extensively about numerical solution of stochastic differential in detail. Platen [10] buttressed this with the discrete time strong and weak approximation methods for the numerical methods to get the solution of stochastic differential equations.Higham [4] contributed and solve the approximate solution of SDEs with few problems. Higham and kloeden [5] further work on nonlinear stochastic differential equation as they presented two explicit methods for $It\hat{o}$ SDEs with Poisson-driven jumps. Nayak and Chakraverty [6] worked on numerical solution of fuzzy stochastic differential equation, where they review the alternative approach to solve uncertain SDE.
As more realistic, Mathematical Models become required to take into account random effects and influences in real world systems, SDEs have become essential in the accurate description of such situations [2]. The solutions are continuous-time stochastic processes and methods for the computational solution of stochastic differential equation are based on similar techniques for stochastic dynamic [3].

We consider a general SDE, which when given in symbolic differential form in one dimension is

$$dX_t = a(X_t)dt + b(X_t)dW_t\ ,\ X(0) = X_0,\ \ 0 \le t \le T \qquad (1)$$

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where $a(X_t)$ is the drift parameter, $b(X_t)$ is the diffusion parameter(or noise term) and W_t is a Wiener process. If the diffusion parameter does not depend on X_t, we say the equation has additive noise, otherwise the equation has a multiplicative noise. A Wiener process $W = W_t, 0 \le t \le T$ is a Gaussian process that depends continuously on time such that

1. $W(0) = 0$(with probability one)
2. for $0 \le t \le T$, $E\big(W(t)\big) = 0$ and for $0 \le t \le T$, $Var\big(W(t) - W(s)\big) = t - s$
3. for $0 \le s < t < u < v \le T$, the increments $W(t) - W(s)$ and $W(v) - W(u)$ are independent.

The Wiener process, named after Norbert Wiener, is a mathematical construct that formalizes random behavior characterized by the botanist Robert Brown in 1827 commonly called Brownian motion. The Stochastic integral to equation (1) can be expressed as

$$X_t = X_0 + \int_{t_o}^{t} a(X_t)ds + \int_{t_o}^{t} b(X_t)\, dW_s \qquad (2)$$

wherethe first integral is a regular (Riemann or Lebesgue) integral and the second integral is a stochastic integral, usually interpreted in the $It\hat{o}$ or Stratonovich form.

The dW_s of Brownian Motion W_s is called *White noise*, a typical solution is a combination of drift and diffusion of Brownian motion.

It is important in the case of numerical analysis to have an equation with a known solution so that the accuracy of a numerical scheme can be determined. We there consider a stochastic differential equation which has a multiplicative noise and explicit solution used to model the randomness of underlying asset in financial mathematics often called Black-Scholes diffusion equation as in (1) and has the explicit solution[12]

$$X_t = X_0 exp\left[\left(a - \frac{b^2}{2}\right)t + bW_t\right] \qquad (3)$$

for $t \epsilon [0, T]$ and Wiener process $W = (W_t, t \ge 0)$

II. STOCHASTIC TAYLOR EXPANSION

Much of the deterministic numerical analysis for Ordinary differential equations is based on manipulating and truncating Taylor expansions.

The $It\hat{o}$-*Taylor* expansion is based on repeated iterations of $It\hat{o}$ formula. We shall consider again the integral equation (2). Note that we require the terms *a* and *b* to satisfy a linear growth bound and to be sufficiently smooth. For any twice continuously differentiable function: $\Re \rightarrow \Re$ $It\hat{o}$'s formula gives**.**

$$f(X_t) = f(X_{t_0}) + \int_{t_0}^{t} \left(a(X_s)f'(X_s) + \tfrac{1}{2}b^2(X_s)f''(X_s)\right) ds +$$

$$\int_{t_0}^{t} b(X_s)f'(X_s)dW_s \qquad (4)$$

Using the operators L^0 and L'

$$L^0 f = af' + \tfrac{1}{2}b^2 f'' \text{ and } L' = bf' \qquad (5)$$

equation (4) gives

$$f(X_t) = f(X_{t_0}) + \int_{t_0}^{t} L^0 f(X_s)ds + \int_{t_0}^{t} L' f(X_s)dW_s$$

If we apply the relation (5) to the functions $f = a$ and $f = b$, we have

$$X_t = X_{t_0} + \int_{t_0}^{t} (a(X_{t_0}) + \int_{t_0}^{s} L^0 a(X_s) dz + \int_{t_0}^{s} L^{'} a(X_z) dW_z) ds$$

$$+ \int_{t_0}^{t} (b(X_{t_0}) + \int_{t_0}^{s} L^0 b(X_s) dz + \int_{t_0}^{s} L^{'} b(X_z) dW_z) ds$$

$$= X_{t_0} + a(X_{t_0}) \int_{t_0}^{t} ds + b(X_{t_0}) \int_{t_0}^{t} dW_s + R_1 \qquad (6)$$

Where $R_1$is the remainder term.

$$R_1 = \int_{t_0}^{t}\int_{t_0}^{s} L^0 a(X_z) dz\, ds + \int_{t_0}^{t}\int_{t_0}^{s} L^{'} a(X_z) dW_z ds + \int_{t_0}^{t}\int_{t_0}^{s} L^0 b(X_z) dz\, dW_s + \int_{t_0}^{t}\int_{t_0}^{s} L^{'} b(X_z) dW_z dW_s \qquad (7)$$

Using$f = L'b$ in (6)

$$X_t = X_{t_0} + a(X_{t_0}) \int_{t_0}^{t} ds + b(X_{t_0}) \int_{t_0}^{t} dW_s + L^{'} b(X_{t_0}) + \int_{t_0}^{t}\int_{t_0}^{s} dW_z dW_s + R_2$$

$$= X_{t_0} + a(X_{t_0}) \int_{t_0}^{t} ds + b(X_{t_0}) \int_{t_0}^{t} dW_s + b(X_{t_0}) b'(X_{t_0}) \int_{t_0}^{t}\int_{t_0}^{s} dW_z dW_s + R_3$$

with remainder

$$R_3 = \int_{t_0}^{t}\int_{t_0}^{s} L^0 a(X_z) dz\, ds + \int_{t_0}^{t}\int_{t_0}^{s} L^{'} a(X_z) dW_z ds + \int_{t_0}^{t}\int_{t_0}^{s} L^0 b(X_z) dz\, dW_s$$

$$+ \int_{t_o}^{t}\int_{t_o}^{s}\int_{t_o}^{z} L^o L^{'} b(X_u) du dW_z dW_s + \int_{t_o}^{t}\int_{t_o}^{s}\int_{t_o}^{z} L^{'} L^{'} b(X_u) dW_u dW_z dW_s$$

It can further be expressed with the multiple Itô integral holding on already apparent in the preceding example as

$$\int_{t_o}^{t} ds, \quad \int_{t_o}^{t} dW, \quad \int_{t_0}^{t}\int_{t_0}^{s} dW_z dW_s$$

This has proven to be a very useful tool in both theoretical and practical investigations, particularly in numerical analysis. It allows the approximation of a sufficiently smoothfunction in a neighborhoods of a given point to any desired order of accuracy. [3][8]

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III.STRONG ORDER 1.5 TAYLOR SCHEME

The Euler-Maruyama and Milstein Scheme can be considered to be specific cases of a more general class of methods know as strong Taylor schemes or approximations form by including approximately many terms from stochastic-Taylor expansions.

We consider the Taylor order 1.5 scheme for SDE (1) as

$$Y_{n+1} = Y_n + a(Y_n)\Delta + b(Y_n)\Delta W + \tfrac{1}{2}b(Y_n)b'(Y_n)((\Delta W)^2 - \Delta) + b(Y_n)a'(Y_n)\Delta Z$$

$$+\tfrac{1}{2}\Delta^2\left(a(Y_n)a'(Y_n) + \tfrac{1}{2}b^2(Y_n)a''(Y_n)\right) + \left(a(Y_n)b'(Y_n) + \tfrac{1}{2}b^2(Y_n)b''(Y_n)\right)$$

$$(\Delta W\Delta - \Delta Z) + b(b(Y_n)b''(Y_n) + (b(Y_n))^2) + \left(\tfrac{1}{3}(\Delta W)^2 - \Delta\right)\Delta W \qquad (8)$$

It becomes clear whether the higher-order methods are needed in a given application depends on how the resulting approximate solutions are to be used. we consider a particular Brownian path and compute for successively $\Delta = 2^{-2}$, 2^{-4} produces the plots below.

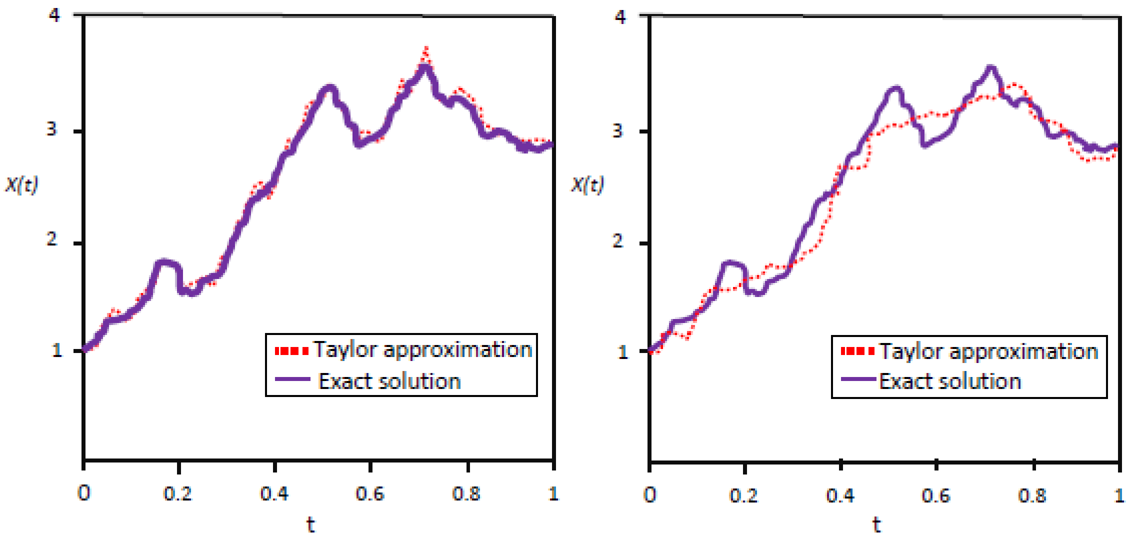

Figure 1. The comparison of the Taylor scheme with the exact solution where $a = 2, b = 1$ at $\Delta = 2^{-2},\ 2^{-6}$.

IV.EULER-MARUYAMA

One of the simplest examples of strong approximations is the Euler or Euler-Maruyama method. We consideranItô process

$X = \{ X_t , t_0 \leq t \leq T\}$Satisfying the scalar stochastic differential equation (1)

$$dX_t = a(X_t)dt + b(X_t)dW_t (9)$$

on$t_0 \leq t \leq T$ with the initial value $X_{t_0} = X_0$ for a given discretization $t_0 \leq \tau_0 < \tau_1 < \ldots < \tau_n = T$ of the time interval $[t_0, T]$, the Euler or Euler-Maruyama approximation is a continuous time stochastic process $Y = \{Y(t), t_0 \leq t \leq T\}$ satisfying the iterative scheme.

$$Y_{n+1} = Y_n + a(Y_n)\Delta + b(Y_n)\Delta W \quad (10)$$

for$n = 0,1,2,\ldots, N-1$ with initial $Y_0 = X_0$ for $\Delta W_n = W_{\tau_{n+1}} - W_{\tau_n}$. From the definition of Wiener process, it is follows that these increments are independent Gaussian random variable with mean $E(\Delta W_n) = 0$ and variance $E(\Delta W_n)^2) = \Delta$.

In examine the first terms of stochastic Taylor expansion, we see that these form a basis of the Euler-Maruyama scheme upon evaluating the integrals, forming Δ and ΔW respectively. When the diffusion parameter $\equiv 0$, it reduces to ordinary deterministic Euler Scheme. The Euler-Maruyama converges with strong order $k = \frac{1}{2}$ while Euler converges with strong order $k = 1$. [8][11]

V.PATHWISE APPROXIMATION AND STRONG CONVERGENCE

The concept of strong convergence uses the concept of the absolute error, which is the expectation of the absolute value of the difference between the approximation and the Itô process at the time T, that is

$$\in= (E|X_T - Y(T)|^q)^{\frac{1}{q}} \quad (11)$$

for some $q \geq 1$ and gives a measure of the pathwise closeness at the end of the time interval $[0, T]$. We say that a discrete time approximation $Y$with step size δ converges strongly to X at time T if

$$\lim_{\delta \downarrow 0} E(|X_T - Y(T)|) = 0 \quad (12)$$

Where Y is the approximate solution computed with constant stepsizeδ and E denotes expected value. For strongly convergent approximations, we further quantify the rate of convergent by the concept of order. An SDE convergence in the stepsize with order $k > 0$ at time T if there exist positive constant C, which does not depend on δ and a $\delta_0 > 0$ such that

$$C(\delta) = E(|X_T - Y(T)|) \leq C\delta^k \quad (13)$$

for sufficiently stepsize δ. This definition generalizes the standard convergence caterionfor ordinary differential equations. Although the Euler method for ordinary differential equations has order 1, while the strong order of Euler-Murayama method for SDE is $1/2$. This fact was proved in Giklman and Skorokhod(1972)[9]

Now that we have the necessary conditions in place, from equation (1), using the same parameter as in figure 1, we also consider a

Brownian path and compute the Euler-Maruyama solution for two step size, taking successively $\Delta = 2^{-2},\ 2^{-4}$ produces the plots below. As we expect, the solution become less accurate as we increase the step size δt of the method

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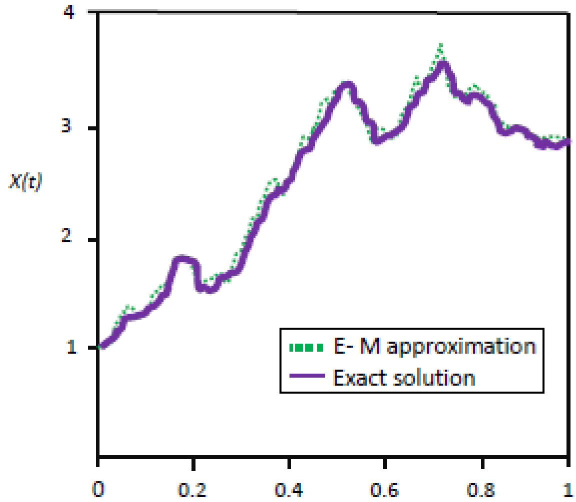

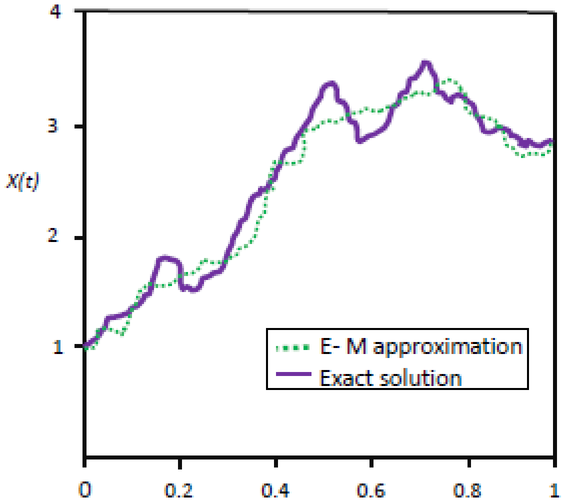

Figure 2. The comparison of the Euler-Maruyama scheme with the exact solution where $a = 2, b = 1$ at $\Delta = 2^{-2}$, 2^{-6}

VI.WEAK EULER SCHEME

Strong convergence allows accurate approximations to be compute on an individual realization basis. If, for example one only requires to compute a moment of solution X, we are not required to approximate individual path of X which leads to the concept of weak convergence. We say that a discrete time approximation Y of a solution X of an SDE converges in the weak sense as $\delta \downarrow 0$ with respect to a class C of test function $g: \Re^d \to \Re$ if we have

$$\lim_{\delta \downarrow 0} |E(g(X_T) - E(g(Y(T)))| = 0 \qquad (14)$$

for all $g\epsilon C$. If C contains all polynomials, this definition implies the convergence of all moments which will involve the investigation of all moments. We shall say that a time discrete approximation Y converges weakly with order $k > 0$ to X at time T as $\delta \downarrow 0$ if for each polynomial g, there exists a positive constant C, which does not depend on δ and a finite $\delta > 0$ such that

$$|E(g(X_T) - E(g(Y(T)))| \leq C\delta^k \qquad (15)$$

for each $\delta \epsilon (0, \delta_0)$. The Euler approximation usually converges with weak order $k > 0$ in contrast with the strong order $k = 1/2$. [3][8][12]

VII.MILSTEIN METHOD

Applying the Stochastic *Itô-Taylor* expansion

$$b(X_{t_0})b'(X_{t_0}) \int_{t_o}^{t} \int_{t_o}^{s} dW_z ds = b(X_{t_0})b'(X_{t_0}) I_{(1,1)}$$

$$= b(X_{t_0})b'(X_{t_0}) \frac{1}{2}((\Delta W)^2 - \Delta)$$

We obtain the Milstein scheme.

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$$Y_{n+1} = Y_n + a(Y_n)\Delta_n + b(Y_n)\Delta W_n + b\left(X_{t_0}\right)b'\left(X_{t_0}\right)\frac{1}{2}((\Delta W)^2 - \Delta) \qquad (17)$$

which has strong order of convergence one (1). Note that the Milstein method is identical to the Euler- Maruyama method if $X \equiv 0$ in the diffusion part $b(X,t)$ of the equation. Under this condition, Milstein will in general converges to the correct Stochastic Solution process faster than Euler-Maruyama as the step size δt goes to zero.[11s]. From equation (1), applying the same parameters used in figure 1, gives the graphical solution of Milstein approximation against exact solution

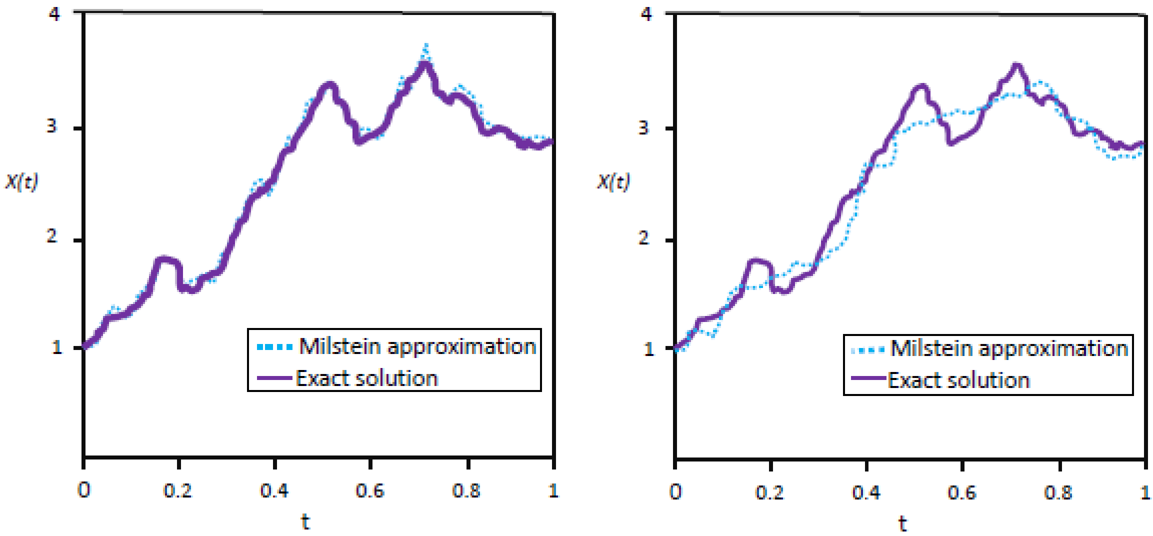

Figure 2. The comparison of the Milstein scheme with the exact solution where $a = 2, b = 1$ at $\Delta = 2^{-2},\ 2^{-6}$

We therefore consider the solution of the convergence for Euler-Murayama and Milstein as we infer the value of the constant k in (14) above by plotting the log of mean error of a series of experiments of the log of the step size, δt. The value of k will be the slope at $\Delta t = 2^{-8}$ of approximate solution of (1) for the error scales$\Delta t^{\frac{1}{2}}$ and Δt for Euler-Maruyama and Milstein respectively

Δt	Euler-Maruyama	Milstein
2^{-1}	0.15825	0.05275
2^{-2}	0.12555	0.02478
2^{-3}	0.07507	0.00685
2^{-4}	0.05950	0.00725
2^{-5}	0.03771	0.00304
2^{-6}	0.02458	0.00104
2^{-7}	0.01316	0.00087

2^{-8}	0.01528	0.00036
2^{-9}	0.01078	0.00013
2^{-10}	0.00680	0.00002

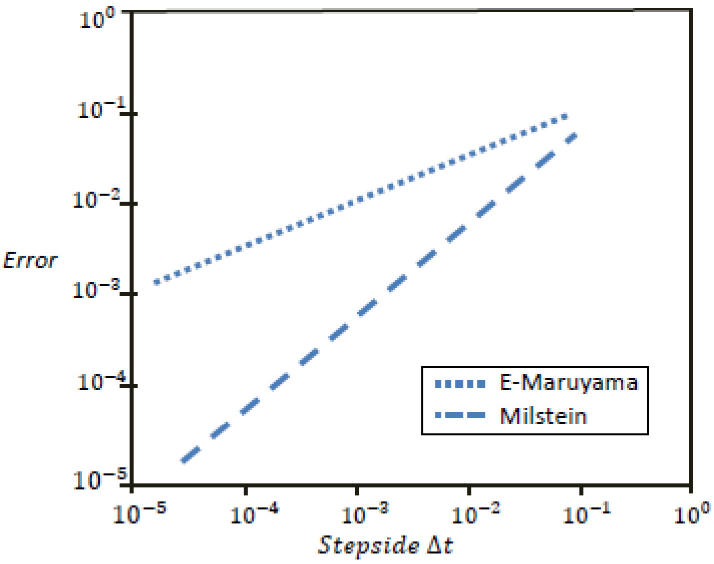

Figure 4. Error in the Euler- Maruyama and Milstein methods

VIII. SUMMARY AND CONCLUSION

This paper has discussed two three techniques for exploring the behavior of stochastic differential equation, taking into consideration the Brownian Motion which served as a basis in finance for computing the expected path of a function of stochastic process.

We developed some numerical techniques for solving stochastic differential equation (SDEs) such as the Euler-Maruyama, Milstein Taylor methods.

We finally performed some convergences analysis and found that Milstein was the better performer in this respect, especially while considering strong convergence.

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