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Some Properties of the Two-Dimensional Kinematic Surfaces Obtained by an Equiform Motion of a Sinusoidal Curve

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ABSTRACT: In this paper we consider an equiform motion of a sinusoidal curve in the Euclidean space E^5. We are studying locally and analyze corresponding the two-dimensional kinematic surfaces when the scalar curvature K is constant. We describe the equations and give examples that govern such the surfaces.

KEYWORDS: Kinematic surface, Equiform motions, Scalar curvature, Sinusoidal curve.

I. INTRODUCTION

The sine curve or sinusoid is a mathematical curve that describes a smooth repetitive oscillation. It is named after the function sine, of which it is the graph. It occurs often in pure and applied mathematics, as well as physics, engineering, signal processing and many other fields. The sine wave is important in physics because it retains its wave shape when added to another sine wave of the same frequency and arbitrary phase and magnitude. It is the only periodic waveform that has this property. This property leads to its importance in Fourier analysis and makes it acoustically unique.

In the n-dimensional Euclidean space E^n, an affine transformation whose linear part is composed by an orthogonal and a homothetical transformation is called an equiform transformation [8, 13, 14, 15, 16, 4, 10]. Such an equiform transformation maps points $x \in E^n$ according to the rule

$$x \to s\,A\,x + d, \qquad A \in SO(n), s \in R^+, d \in E^n. \tag{1}$$

The positive number s represents the scaling factor. Eq. (1) defines an equiform motion if its parameters, including s, are given as functions of the time variable t. Hence, a smooth one-parameter equiform motion moves a point x via the relation $x(t) = s(t)A(t)x(t) + d(t), t \in R^+$, the kinematic corresponds to this transformation group is called equiform kinematic [5, 7].

Under the assumption of the constancy of the scalar curvature, kinematic surfaces studied by the motion of a circle have been obtained in [3]. Moreover, the hypersurfaces in space forms generated by one-parameter family of spheres having constant curvature are considered in [6, 9, 11, 12].

In this paper, we consider the equiform motions of a sinusoidal curve c_0 in E^n. The point paths of the sinusoidal curve generate a two dimensional kinematic surface X, containing the position of the starting sinusoidal curve c_0. At any moment, the infinitesimal transformations of the motion will map the points of the sinusoidal curve c_0 into the velocity vectors whose end points will form an affine image of c_0 that will be, in general, a sinusoidal curve in the moving space Σ. Both curves are planar and therefore, they span a subspace W of R^n, with $\dim(W) < 5$. This is the reason why we restrict our considerations to dimension $n = 5$.

Let $x(\varphi)$ and $X(t,\varphi)$ denote the parameterization of c_0 and the resultant two-dimensional kinematic surface by the equiform motions, respectively, we can consider a certain position of the moving space given by $t = 0$, and obtain

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information about the motions, at least during a certain period around $t = 0$, if its characteristics for one instant is given. The purpose of this paper is to describe the scalar curvature K of the two-dimensional kinematic surfaces obtained by the motion of a sinusoidal curve locally.

II. Locally Representation of the Motion

In two copies Σ^0, Σ of Euclidean 5-space E^5, we consider a sinusoidal curve c_0 in the x_1x_2-plane of Σ^0 and represented by

$$x(\varphi) = (a\,\varphi, sin\varphi, 0,0,0)^T, \qquad \varphi \in [0,2\pi].$$

The general representation of the equiform motion in E^5 of this curve is given by:

$$X(t,\varphi) = s(t)A(t)x(\varphi) + d(t), \qquad t \in I \subset R. \qquad (2)$$

Here, $d(t) = (b_1(t)\ , b_2(t)\,, b_3(t), b_4(t), b_5(t))^T$ is the position of the origin at time t, $A(t) = (\,a_{ij}(t)\,)\,(\,i\,, j = 1, \dots, 5\,)$ is an orthogonal matrix and $s(t)$ is the scaling factor of the moving system. Using Taylor's expansion up to the first order, the representation of the motion is given by:

$$X(t,\varphi) = [s(0)A(0)[\,s(0)A(0) + \ s(0)A^{\dot{}}(0)]t]x(\varphi) + \ d(0) + \ t\,d^{\dot{}}(0).$$

Where "." denotes the differentiation with respect to the time variable t. As an equiform motion has an invariant point, we can assume without loss of generality that the moving frame Σ^0 and the fixed frame Σ coincide at the zero position $t = 0$. Then we have $A(0) = I, s(0) = 1$. and $d(0) = 0$. Thus

$$X(t,\varphi) = \ [I \ + \ (\,s'I \ + \Omega\,)t]x(\varphi) + \ t\,d'.$$

Where $\Omega \ = \ A^{\cdot}(0) \ = \ \omega_k, k \ = \ 1,2,3, \dots, 10, d' = \ d^{\cdot}(0), s' \ = \ s^{\cdot}(0)$ and the representation of the motion up to the first order is given by:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix}(t,\varphi) = \begin{bmatrix} 1+s't & t\omega_1 & t\omega_2 & t\omega_3 & t\omega_4 \\ -t\omega_1 & 1+s't & t\omega_5 & t\omega_6 & t\omega_6 \\ -t\omega_2 & -t\omega_5 & 1+s't & t\omega_1 & t\omega_1 \\ -t\omega_3 & -t\omega_6 & -t\omega_8 & 1+s't & t\omega_1 \\ -t\omega_4 & -t\omega_7 & -t\omega_9 & -t\omega_{10} & 1+s't \end{bmatrix} \begin{bmatrix} a\varphi \\ sin\varphi \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} b_1' \\ b_2' \\ b_3' \\ b_4' \\ b_5' \end{bmatrix},$$

or in the equivalent form

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} 1+s't \\ -t\omega_1 \\ -t\omega_2 \\ -t\omega_3 \\ -t\omega_4 \end{bmatrix} a\varphi + \begin{bmatrix} t\omega_1 \\ 1+s't \\ -t\omega_5 \\ -t\omega_6 \\ -t\omega_7 \end{bmatrix} sin\varphi + t \begin{bmatrix} b'_1 \\ b'_2 \\ b'_3 \\ b'_4 \\ b'_5 \end{bmatrix}, \qquad (3)$$

For any fixed t in the above expression (3), we generally get a sinusoidal curve with its intersection pointt $(b_1', b_2', b_3', b_4', b_5')$ subject to the following condition

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$$\omega_2\omega_5 + \omega_3\omega_6 + \omega_4\omega_7 = 0 \qquad (4)$$

III.SCALAR CURVATURE OF THE TWO DIMENSIONAL KINEMATIC SURFACE

In this section, we give a calculating formula of the scalar curvature of a sinusoidal curve surfaces in E^5 generated by an equiform motions. The tangents to the parametric curve t = const, φ = const of this surface (2), are represented by:

$$X_t(t,\varphi) = (s^{'}I + \Omega)x(\varphi) + \, d^{'}, \qquad X_\varphi(t,\varphi) = [I + (s'I + \Omega)t]\, x'(\varphi).$$

The first fundamental quantities of a sinusoidal surface are given by:

$$g_{11} = X_t^T\, X_t\,, \qquad g_{12} = X_\varphi^T\, X_t, \qquad g_{22} = X_\varphi^T\, X_\varphi.$$

A straightforward computation leads to the coefficients of the first fundamental form defined by

$$g_{11} = [(s^{'}I - \Omega)x^T(\varphi) + d^{'T}][(s'I + \Omega)x(\varphi) + d'],$$
$$g_{12} = x^{'T}(\varphi)[(s^{'}I + \Omega)x(\varphi) + d^{'}],$$
$$g_{22} = x^{'T}(\varphi)x'(\varphi).$$

Thus

$$g_{11} = \alpha + \beta\varphi + \gamma\varphi^2 + \sigma\, sin\,\varphi - \delta cos\, 2\varphi,$$
$$g_{12} = a\, b_1^{'} + \tfrac{1}{2}\beta\, t + (a^2\, s^{'} + t\,\gamma)\varphi + \left(b_2^{'} + \tfrac{1}{2} t\,\sigma - a\,\omega_1\varphi\right) cos\,\varphi + \, a\omega_1 sin + \left(\tfrac{1}{2} s^{'} + t\right) sin\, 2\varphi, \qquad (5)$$
$$g_{22} = \left(\frac{1}{2} + a^2\right)(1 + 2s^{'}t) + (\gamma + \delta)t^2 + \left(\frac{1}{2} + s^{'}t + t^2\delta\right) cos\, 2\varphi.$$

Where

$$\alpha = \textstyle\sum_{i=1}^{5} b_i'^2 \; + \tfrac{1}{2}(s^{'2} + \omega_1^2 + \sum_{i=5}^{7}\omega_i^2),$$

$$\beta = 2a(b_1^{'}\, s' - \textstyle\sum_{i=1}^{4} b_{i+1}^{'}\omega_i),$$

$$\gamma = a^2(s^{'2} + \textstyle\sum_{i=1}^{4}\omega_i^2\,),$$

$$\delta = \tfrac{1}{2}(s^{'2} + \omega_1^2 + \textstyle\sum_{i=5}^{7}\omega_i^2), \qquad (6)$$

$$\sigma = 2(b_2^{'}s' \, + b_1^{'}\omega_1 - \textstyle\sum_{i=3}^{5} b_i^{'}\omega_{i+1}).$$

The scalar curvature of $x(t,\varphi)$ is defined by

$$K = \sum_{i,j,l=1}^{2} [g^{ij}\,[\frac{\partial\Gamma_{ij}^l}{\partial x_l} - \frac{\partial\Gamma_{il}^l}{\partial x_j} + \sum_{i,j,l=1}^{2} (\Gamma_{ij}^l\,\Gamma_{lm}^m - \Gamma_{il}^m\,\Gamma_{jm}^l\,)],$$

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where

$$\Gamma_{ij}^{k} = \frac{1}{2}\sum_{m=1}^{2} g^{km} \left(\frac{\partial g_{im}}{\partial x_j} + \frac{\partial g_{jm}}{\partial x_l} - \frac{\partial g_{ij}}{\partial x_m}\right),$$

Christoffel symbols of the second kind where $x_i \in \{i, \varphi\}$, $\{i, j, k\}$ indices that take the value 1 or 2 are and (g^{lm}) is the inverse matrix of (g_{ij}). By using the Mathematica programme, we can obtain explicit computation formula of the scalar curvature K respectively:

$$K = \frac{P(\varphi^m \cos n\varphi, \varphi^m \sin n\varphi)}{Q(\varphi^m \cos n\varphi, \varphi^m \sin n\varphi)} = \frac{\sum_{n=0}^{4}\sum_{m=0}^{2}\left(A_{n,m}\varphi^m \cos n\varphi + B_{n,m}\varphi^m \sin n\varphi\right)}{\sum_{n=0}^{8}\sum_{m=0}^{4}\left(E_{n,m}\varphi^m \cos n\varphi + F_{n,m}\varphi^m \sin n\varphi\right)}. \qquad (7)$$

The assumption of the constancy of the scalar curvature K implies that (7) converts into

$$KQ(\varphi^m \cos n\varphi, \varphi^m \sin n\varphi) - P(\varphi^m \cos n\varphi, \varphi^m \sin n\varphi) = 0. \qquad (8)$$

We can write the equation (8) as a linear combination of the functions $\{\varphi^m \cos n\varphi, \varphi^m \sin n\varphi\}$ namely, $\sum_{n=0}^{4}\sum_{m=0}^{2}\left(C_{n,m}\varphi^m \cos n\varphi + D_{n,m}\varphi^m \sin n\varphi\right)$, the corresponding coefficients must equal zero. We will analyze the cases $K = 0$ and $K \neq 0$.

V. TWO-DIMENSIONAL KINEMATIC SURFACE WITH $K = 0$

In this subsection we assume that $K = 0$ on the surface $X(t, \varphi)$. From (7), we have

$$\sum_{n=0}^{4}\sum_{m=0}^{2}\left(A_{n,m}\varphi^m \cos n\varphi + B_{n,m}\varphi^m \sin n\varphi\right) = 0.$$

Then the work consists in the explicit computations of the coefficients A_n and B_n. We distinguish different cases that fill all possible cases. The coefficient $A_{1,2}$ is

$$A_{1,2} = 6as'(\gamma - 2a^2\delta)\omega_1 = 0.$$

But $a \neq 0$ and $s' \neq 0$, it follows that $\omega_1 = 0$ or $\gamma = 2a^2\delta$.

(1) At $\omega_1 = 0$. The coefficient $B_{4,1} = -\gamma(s'^2 - 2\delta) = 0$, this leads to $\gamma = 0$ or $\delta = \frac{1}{2}s'^2$.

(a) If $\gamma = 0$ from expression (6), we have a contradiction.

(b) If $\delta = \frac{1}{2}s'^2$ from expression (6), we have $\omega_i = 0, i = 5,6,7$ and $\sigma = 2b_2's'$. The coefficient

$$A_{2,2} = 2a^2s'^2(a^2s'^2 - \gamma) = 0.$$

Then $\gamma = a^2s'^2$, from expression (6) we get $\omega_i = 0, i = 2,3,4$.

(2) At $\gamma = 2a^2\delta$ from expression (6), we have

$$\omega_2^2 + \omega_3^2 + \omega_4^2 = \omega_5^2 + \omega_6^2 + \omega_7^2,$$

and the coefficient $B_{3,2} = a^2\delta(s'^2 - 2\delta + \omega_1^2)$this gives two possibilities $\delta = 0$ or $\delta = \frac{1}{2}(s'^2 + \omega_1^2)$.

(a) If $\delta = 0$ from expression (6) we have a contradiction.

(b) If $\delta = \frac{1}{2}(s'^2 + \omega_1^2)$, from expression (6) leads to $\omega_i = 0, i = 2 \leq i \leq 7, \beta = 2a(b_1's' - b_2'\omega_1)$ and

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$\sigma = 2(b_2' s' + b_1' \omega_1)$. The coefficient

$$A_{1,0} = \omega_1^2(1+2a^2)(2b'^2_1 + 2b'^2_2 + s'^2 - 2\alpha + \omega_1^2) = 0.$$

Then $\omega_1 = 0$or $\alpha = b'^2_1 + b'^2_2 + \frac{1}{2}(s'^2 + \omega_1^2)$, from expression (6) we have $b_3' = b_4' = b_5' = 0$. Hence, we conclude the following theorem.

Theorem.1 Let $X(t,\varphi)$ be a two-dimensional kinematic surface obtained by the equiform motions of the sinusoidal curve c_0 and given by (2) under condition (4). Then $K = 0$ on this surface if and only if one of the following conditions satisfies:

(i) $\omega_i = 0; \quad 2 \leq i \leq 7, \; b_j' = 0; \quad j = 3,4,5.$

(ii) $\omega_i = 0, \quad 1 \leq i \leq 7.$

VI. TWO-DIMENSIONAL KINEMATIC SURFACE WITH $K \neq 0$

In this subsection we assume that the scalar curvature K of the sinusoidal surface $X(t,\varphi)$ obtained by the equiform motions of a sinusoidal curve and given by (2) under condition (4) is a non-zero constant. The identity (8) writes then as

$$\sum_{n=0}^{8}\sum_{m=0}^{4}\left(C_{n,m}\varphi^m \cos n\varphi + D_{n,m}\varphi^m \sin n\varphi\right) = 0.$$

Following the same scheme as in the case $K = 0$ studied in subsection (IV), we begin to compute the coefficients $C_{n,m}$ and $D_{n,m}$. Let us put $t = 0$. The coefficient

$$C_{3,3} = \frac{1}{4}K(\gamma - a^2\omega_1^2)^2 = 0.$$

But $K \neq 0$, then $\gamma = a^2\omega_1^2$ from expression (6) we have a contradiction.

Theorem 2. There are not a two-dimensional kinematic surfaces obtained by the motion of a sinusoidal curve c_0 and given by (2) under condition (4) whose scalar curvature K is a non-zero constant.

Corollary 1. Let $X(t,\varphi)$ be a two-dimensional kinematic surface obtained by the motion of a sinusoidal curve c_0 and given by (2) under condition (4). If the scalar curvature K is constant then $K = 0$.

VII. EXAMPLES OF A TWO – DIMENSIONAL KINEMATIC SURFACE WITH VANISHING SCALAR CURVATURE

In this section, we illustrate the previous results by giving two examples of a two-dimensional kinematic surface $X(t,\varphi)$ with constant scalar curvature $K = 0$. The first example corresponds with the case $b_1'b_2' \neq 0$. In the second example, we put $b_1' = 0, b_2' = 0$.

Example1. Case $b_1'b_2' \neq 0$.

Consider the following orthogonal matrix.

$$A(t) = \begin{bmatrix} cost & sint & 0 & 0 & 0 \\ -sint & cost & 0 & 0 & 0 \\ 0 & 0 & \cos^2 t & -sint & sint\,cost \\ 0 & 0 & sint & cost & 0 \\ 0 & 0 & -sint & 0 & cost \end{bmatrix}$$

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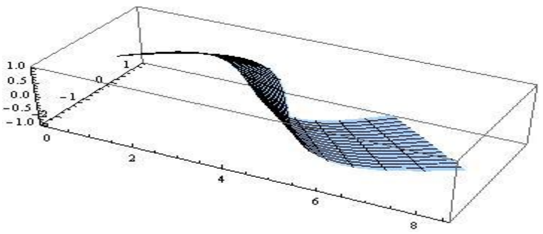
we assume that the factor $s(t) = e^t$ and $d(t) = (t, t, 0,0,0)^T$. Here we have $\omega_1 = 1$, $\omega_8 = \omega_9 = 1$, and $\omega_i = 0$ for $i = 2,3, \dots ,7,10$, $s' = 1, b_1' = b_2' = 1$ and $b_i' = 0$, for $i = 3,4,5$. Then Theorem 1 says us that the corresponding surface $X(t, \varphi)$ has $K = 0$. In Figure 1, we display a piece of $X(t, \varphi)$ of Example 1 in axonometric viewpoint $Y(t, \varphi)$. For this, the unit vectors $E_4 = (0,0,0,1,0)$ and $E_5 = (0,0,0,0,1)$ are mapped onto the vectors (1,1,0) and (0,1,1) respectively [8]. Then

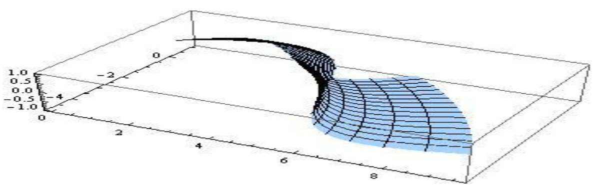
$$X(t,\varphi) = \begin{bmatrix} t \\ t \\ 0 \\ 0 \\ 0 \end{bmatrix} a\varphi + \begin{bmatrix} 1+t \\ -t \\ 0 \\ 0 \\ 0 \end{bmatrix} sin\varphi + t \begin{bmatrix} t \\ 1+t \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

and

$$Y(t,\varphi) = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} a\varphi + \begin{bmatrix} 1+t \\ -t \\ 0 \end{bmatrix} sin\varphi + t \begin{bmatrix} t \\ 1+t \\ 0 \end{bmatrix}.$$



(a)



(b)

Figure 1: In (a), we have a piece of a kinematic surface in axonometric view $Y(t, \varphi)$ with zero scalar curvature $K = 0$; in (b) we have the corresponding surface $X(t, \varphi)$ with equation (1) that approximates.

Example 2. Case $b_1' = b_2' = 0$. Let now the orthogonal

$$A(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & cost & sint & 0 \\ 0 & 0 & -sint & \cos^2 t & sint\,cost \\ 0 & 0 & 0 & -sint & cost \end{bmatrix}.$$

We assume $s(t) = e^t$and $d(t) = (0,0, t, t, t)^T$. Then

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$$s' = 1, \qquad \omega_8 = \omega_{10} = 1, \qquad \omega_k = 0, \qquad k = 1,2,3,4,6,5,7,9$$

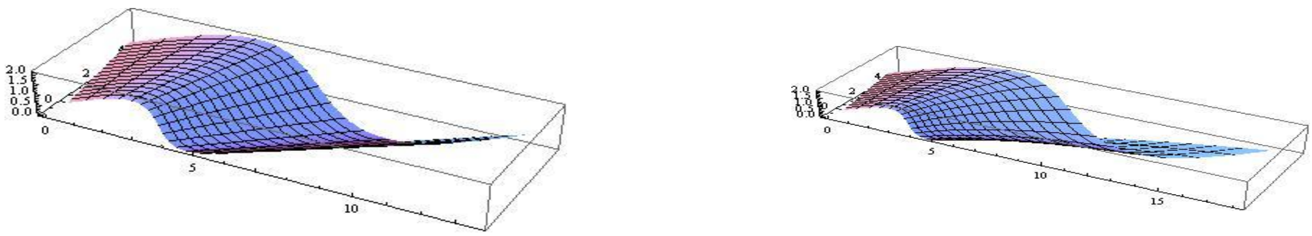
$$b_1' = b_2' = 0; \qquad b_3' = b_4' = b_5' = 1.$$

Theorem 1 says that $K = 0$. We display a piece of $X(t,\varphi)$ of Example 2 in axonometric viewpoint $Y(t,\varphi)$.

$$X(t,\varphi) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} a\varphi + \begin{bmatrix} 1+t \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} sin\varphi + t \begin{bmatrix} 0 \\ 1+t \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

and

$$Y(t,\varphi) = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} a\varphi + \begin{bmatrix} 1+t \\ 0 \\ 0 \end{bmatrix} sin\varphi + t \begin{bmatrix} 0 \\ 1+t \\ 0 \end{bmatrix}.$$



(a) **(b)**

Figure 2 In (a), we have a piece of the two dimensional kinematic surface in axonometric view $Y(t,\varphi)$ with zero scalar curvature $K = 0$; in (b) we have the corresponding surface $X(t,\varphi)$ with equation (1) that approximates.

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