ISSN 2350 - 0328

International Journal of Advanced Research in Science, Engineering and Technology

Vol. 1, Issue 3, October 2014

Exact Solution of nonlinear time-fractional Biological population Equation

Salih M. Elzaki

Math.Dept,Sudan University of Science and Technology, Kingdom of Saudi Arabia

ABSTRACT: This paper proposes a modified version of the differential transform method, which is termed the projected differential transform method (PDTM) .This method involves less computational work and can, thus, be easily applied to initial value problems. (PDTM) is used to determine the exact solutions of some nonlinear time-fractional partial differential equations. A number of illustrative examples are provided and compared with the other methods. The numerical results obtained by these examples are found to be the same.

KEYWORDS: biological population equation, nonlinear partial differential equation- the projected differential transforms method, Numerical method.

I. Introduction

In recent years it turned out that many phenomena in viscous elasticity, fluid mechanics can be successfully modeled by the use of fractional order derivatives. These applications include the fields of biology, chemistry, acoustics control theory, psychology and other areas of science This is due to the fact that, a realistic modeling of a physical phenomenon having dependence not only at the time instant, but also the previous time history can be successfully achieved by using fractional calculus.

Such calculus can be designated as non-integer order of calculus and its subject matter can be traced back to the genesis of integer order differential calculus itself. Though G.W. Leibniz made some remarks on the meaning and possibility of fractional derivatives of order $\frac{1}{2}$ in the late seventeenth century, a rigorous investigation was first carried out by Liouville in a series of papers from 1832 to 1837, where he first defined an operator of fractional integration. Today, fractional calculus generates the derivative and anti derivative operations of differential and integral calculus from non-integer orders to the entire complex plane.[1]-[4]

There are several approaches to the generalization of the notion of differentiation to fractional orders e.g Riemann – Liouville. Grunwald – Letnikow. Caputo and generalized functions approach. Riemann-Lioville fractional derivative is mostly employed by mathematicians but this approach is not suitable for real world physical problems since it requires the definition of fractional order initial conditions, which has the advantage of defining integer order initial conditions which have no physically meaning full explanation as yet. Caputo introduced an alternative definition, for fractional order differential equations. Unlike the Riemann – Liouville approach which derives its definition from repeated integration, the Grunwald – Letnikow formulation approaches the problem from the derivative perspective. This approach is mostly used in numerical algorithms.

Moreover, there are several techniques for the solution of fractional differential equations. The most commonly used ones are, Adomian decomposition method (ADM), variational iteration method (VIM), Fractional difference method (FDM) and power series method . Also there are some classical solution techniques. e.g Laplace transform method, fractional Green's function method, Mellin transform method and method of orthogonal polynomials. Among these solution techniques, the power series method is the most transparent method of solution of fractional differential and integral equations.[3]-[12]

There are several definitions of a fractional derivative of order $\alpha > 0$. The two most commonly used definitions are the ones by Riemann – Liouville and Capout. Each definition uses Riemann-Liouville fractional integration and derivations of whole order. The difference between the two definitions lies in the order of evaluation Riemann-Louville fractional integral of order α is defined as [5]-[8]

ISSN 2350 - 0328

International Journal of Advanced Research in Science, Engineering and Technology

Vol. 1, Issue 3, October 2014

$$J_{x_0}^{\alpha} f(X,x)=\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}(x-t)^{\alpha-1} f(X,t)dt, \quad \alpha>0,\ x>0 \qquad (1)$$

$$X=X(x_1,x_2,....,x_n)$$

The next two equations define Riemann – Liouville and caputo fractional derivatives will of order α, respectively

$$D_{x_o}^{\alpha} f(X,x)=\frac{\partial^m}{\partial x^m}\left[J^{m-\alpha} f(X,x)\right] \qquad (2)$$

$$D_{*x_o}^{\alpha} f(X,x)=J^{m-\alpha}\left[\frac{\partial^m}{\partial x^m} f(X,x)\right] \qquad (3)$$

Where $m-1<\alpha>m$ and m $\in$ N .For now, the Caputo fractional derivative will be denoted by $D_{*x_0}^{\alpha}$ to maintain a clear distinction with the Riemann – Liouville fractional derivative. The Caputo fractional derivative first computes an ordinary derivative followed by a fractional integral to achieve the desired order of fractional derivative. The Riemann – Liouville fractional derivative is computed in the reverse order we have chosen to use the Coputo fractional derivative because it allows traditional initial and boundary conditions to include in the formulation of the problem, but for homogeneous initial condition assumption, these two operators coincide[6]-[10]

II. Fractional projected differential transform method

We introduce the fractional projected differential transform method used in the sequent to obtain approximate analytical solutions for a fractional oscillator this method has been developed by Arikoglu and Ozkol as follows.
The fractional differentiation in Riemann – liouville sense is defined by

$$D_{x_0}^{q} f(X,x)=\frac{1}{\Gamma(m-q)}\frac{\partial^m}{\partial x^m}\left[\int_{x_0}^{x}\frac{f(X,t)}{(x-t)^{1+q-m}}dt\right] \qquad (4)$$

For m-1$\leq$ q $\leq$ m, m$\in$ N , x > x_0 let us expand the analytical and continuous function $f(X,x)$ in terms of a fractional power series as follows:

$$f(X,x)=\sum_{k=o}^{\infty} f(X,k\alpha q)(x-x_o)^{\alpha k} \qquad (5)$$

Where α is the order of fractional and $f(X,k)$ is the fractional projected differential transform of $f(X,x)$.
In order to avoid fractional initial and boundary conditions, we define the fractional derivative in the caputo sense. The relation between the Riemann- Liovuille operator and Caputo operator is given by

$$D_{*x_o}^{q} f(X,x)=D_{x_o}^{q}\left[f(X,x)-\sum_{k=o}^{m-1}\frac{f^{(k)}(X,x_o)}{k!}(x-x_o)^k\right] \qquad (6)$$

Setting $f(X,x)=f(X,x)-\sum_{k=o}^{m-1}\frac{f^{(k)}(X,x_o)}{k!}(x-x_o)^k$ in eq (4) and using eq (6) we obtain fractional derivative the caputo sense as follows

ISSN 2350 - 0328

International Journal of Advanced Research in Science, Engineering and Technology

Vol. 1, Issue 3, October 2014

$$D^{q}_{*x_o} f(X,x)=\frac{1}{\Gamma(m-q)}\frac{\partial^m}{\partial x^m}\int_{x_o}^{x}\left[\frac{f(X,t)-\sum_{k=o}^{m-1}\frac{f^{(k)}(X,x_o)}{ki}(t-x_o)^k}{(x-t)^{1+q-m}}\right]dt \qquad (7)$$

Since the initial conditions are implemented to the integer order derivatives, the transformation of the initial conditions are defined as follows:

$$F(X,k)=\begin{cases}\frac{1}{(k/\alpha)!}\left[\frac{\partial^{\frac{k}{\alpha}} f(X,x)}{\partial x^{\frac{k}{\alpha}}}\right]_{x=x_o} & if\ \frac{k}{\alpha}\in z^{+}\ for\ k=0,1,....q(q\alpha-1)\\ 0 & if\ \frac{k}{\alpha}\notin z^{+}\end{cases} \qquad (8)$$

Where q is the order of fractional differential equation considered

Theorems: [9]-[11]

(1) If $f(X,x)=g(X,x)\pm h(X,x)$ Then $f(X,k)=g(X,k)\pm h(X,k)$

(2) If $f(X,x)=c\,g(X,x)$ Then $f(k)=c\,g(k)$, c is constant

(3)If $f(X,x)=g(X,x)h(X,x)$ Then $F(X,k)=\sum_{l=0}^{k}G(X,l)H(X,k-l)$

(4)If $f(X,x)=g_1(X,x)g_2(X,x)..........g_n(X,x)$ Then

$$f(X,k)=\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}....\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}G_1(X,k_1)G_2(X,k_2-k_1)....\times G_{n-1}(X,k_{n-1}-k_{n-2})G_n(X,k-k_{n-1})$$

(5) If $f(X,x)=D^{q}_{x_o}[g(X,x)]$ Then $F(X,k)=\frac{\Gamma(q+1+k/\alpha)}{\Gamma(1+k/\alpha)}G(X,k+\alpha q)$

(6) If $f(X,x)=\left[\frac{\partial^{q_1}}{\partial x^{q_1}}g_1(X,x)\right]\left[\frac{\partial^{q_2}}{\partial x^{q_2}}g_2(X,x)\right]......\left[\frac{\partial^{q_n}}{\partial x^{q_n}}g_n(X,x)\right]$ Then

$$F(X,k)=\sum_{k_{n-1}=o}^{k}\sum_{k_{n-2}=o}^{k_{n-1}}.....\sum_{k_2=o}^{k_3}\sum_{k_1=o}^{k_2}\frac{\Gamma(q_1+1+k_1/\alpha)}{\Gamma(1+k/\alpha)}\frac{\Gamma(q_2+1+(k_2-k_1)/\alpha)}{\Gamma(1+(k_2-k_1)/\alpha}..........\times\frac{\Gamma(q_n+(k_1+k_{n-1})/\alpha)}{\Gamma(1+(k-k_{n-1})/\alpha)}G_1(X,k_1+\alpha q_1)\,G_2(X,k_2-k_1+\alpha q_2).........\times G_n(X,k-k_{n-1}+\alpha q_n)$$

ISSN 2350 - 0328

International Journal of Advanced Research in Science, Engineering and Technology

Vol. 1, Issue 3, October 2014

Where $\alpha, q_i \in z^+$ for $i = 0,1,2,......,n$

(7)If $f(X,x) = (x - x_0)^p$ Then $f(X,k) = \delta(k - \alpha p)$

Where $\delta(X,k) = \begin{cases} 1 & if \quad k = 0 \\ 0 & if \quad k \neq 0 \end{cases}$

III. Analysis of the method

Consider general nonlinear time-fractional biological population equation

$$D_t^\alpha u(X,t) = (u^2)_{xx} + (u^2)_{yy} + f(u) \qquad (9)$$

$$f(u) = hu^a(1 - ru^b) \quad , \quad t > 0, \ x, y \in R \quad 0 < \alpha \leq 2 \ , X = X(x,y)$$

With the initial condition $u(x,y,0) = g(x,y)$

where u denotes population density, f represents the population supply due to births and deaths, h, a, r, b are real numbers, g is given initial condition and D- denotes the differential operator in the sense of Caputo.[2]-[7]

By taking the Projected differential transform method of equation (9)we obtain

$$\frac{\Gamma\left(1+\alpha+\frac{k}{q}\right)}{\Gamma\left(1+\frac{k}{q}\right)} u(X, k+\alpha q) = \left(\sum_{m=0}^{k} u(X,m) u(X,k-m)\right)_{xx} + \left(\sum_{m=0}^{k} u(X,m) u(X,k-m)\right)_{yy} + f\left(u(X,k)\right) \quad , \ k = 0, \alpha q, 2\alpha q, ...$$

$$u(X, k\alpha q) = \frac{\Gamma(1+\alpha(k-1))}{\Gamma(1+\alpha k)} \left[\left(\sum_{m=0}^{\alpha q(k-1)} u(X,m) u(X, \alpha q(k-1) - m)\right)_{xx} + \left(\sum_{m=0}^{\alpha q(k-1)} u(X,m) u(X, \alpha q(k-1) - m)\right)_{yy} + f\left(u(X, \alpha q(k-1))\right) \right]$$

Substituting $u(X, k\alpha q)$ into equation (5) we get

ISSN 2350 - 0328

International Journal of Advanced Research in Science, Engineering and Technology

Vol. 1, Issue 3, October 2014

$$u(X,t)=g(x,y)+\sum_{k=1}^{\infty}\frac{\Gamma(1+\alpha(k-1))}{\Gamma(1+\alpha k)}\left[\left(\sum_{m=0}^{\alpha q(k-1)}u(X,m)u(X,\alpha q(k-1)-m)\right)_{xx}+\right.$$

$$\left.\left(\sum_{m=0}^{\alpha q(k-1)}u(X,m)u(X,\alpha q(k-1)-m)\right)_{yy}+f\left(u\left(X,\alpha q(k-1)\right)\right)\right]t^{\alpha k}$$

IV.APPLICATION

In this section, we present two example with analytical solution to show the efficiency of methods described in the Section III

Example 1: [6] We consider the equation(9) with $a=1, r=0$ and $g(x,y)=\sqrt{xy}$

$$D_t^{\alpha}u(X,t)=(u^2)_{xx}+(u^2)_{yy}+hu \quad ,t>0\ ,\ 0<\alpha\le 2 \qquad (10)$$

$$u(x,y,0)=\sqrt{xy}$$

Using projected differential transform method of equation (10) we have

$$\frac{\Gamma\left(1+\alpha+\frac{k}{q}\right)}{\Gamma\left(1+\frac{k}{q}\right)}u(X,k+\alpha q)=\left(\sum_{m=0}^{k}u(X,m)u(X,k-m)\right)_{xx}+\left(\sum_{m=0}^{k}u(X,m)u(X,k-m)\right)_{yy}$$

$$+hu(X,k) \quad ,\ k=0,\alpha q,2\alpha q,.......$$

$$u(X,k\alpha q)=\frac{\Gamma(1+\alpha(k-1))}{\Gamma(1+\alpha k)}\left[\left(\sum_{m=0}^{\alpha q(k-1)}u(X,m)u(X,\alpha q(k-1)-m)\right)_{xx}+\right.$$

$$\left.\left(\sum_{m=0}^{\alpha q(k-1)}u(X,m)u(X,\alpha q(k-1)-m)\right)_{yy}+hu(X,\alpha q(k-1))\right]$$

$$u(X,\alpha q)=\frac{1}{\Gamma(1+\alpha)}\left[\left(u^2(X,0)\right)_{xx}+\left(u^2(X,0)\right)_{yy}+hu(X,0)\right]=\frac{\sqrt{xy}\,h}{\Gamma(1+\alpha)}$$

ISSN 2350 - 0328

International Journal of Advanced Research in Science, Engineering and Technology

Vol. 1, Issue 3, October 2014

$$u(X,2\alpha q)=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\Big[\ \big(2u(X,0)u(X,\alpha q)\big)_{xx}+\big(2u(X,0)u(X,\alpha q)\big)_{yy}$$
$$+hu(X,\alpha q)\ \Big]=\frac{\sqrt{xy}\,h^{2}}{\Gamma(1+2\alpha)}$$

$$u(X,3\alpha q)=\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\Big[\ \big(u^{2}(X,\alpha q)+2u(X,0)u(X,2\alpha q)\big)_{xx}$$
$$+\big(u^{2}(X,\alpha q)+2u(X,0)u(X,2\alpha q)\big)_{yy}+hu(X,2\alpha q)\ \Big]=\frac{\sqrt{xy}\,h^{3}}{\Gamma(1+3\alpha)}$$

And so on in general

$$u(X,k\alpha q)=\frac{\sqrt{xy}\,h^{k}}{\Gamma(1+k\alpha)}$$

Substituting $u(X,k\alpha q)$ into equation (5) we get

$$u(X,t)=\sum_{k=0}^{\infty}\frac{\sqrt{xy}\,h^{k}}{\Gamma(1+k\alpha)}t^{\alpha k}=\sqrt{xy}\sum_{k=0}^{\infty}\frac{\left(ht^{\alpha}\right)^{k}}{(\alpha k)!}$$

Lim $u(X,t)$ we get

$\alpha\to 2$

$$u(X,t)=\sqrt{xy}\sum_{k=0}^{\infty}\frac{\left(\sqrt{ht}\right)^{2k}}{(2k)!}=\sqrt{xy}\cosh\sqrt{ht}$$

And $\alpha\to 1$ $\quad u(X,t)=\sqrt{xy}\sum_{k=0}^{\infty}\frac{(ht)^{k}}{(k)!}=\sqrt{xy}\,e^{ht}$

Example2: [6]

We consider the equation(9) with $a=1,b=1$ and $g(x,y)=e^{\left(\sqrt{\frac{hr}{8}}\right)(x+y)}$

$$D_t^{\alpha}u(X,t)=\left(u^{2}\right)_{xx}+\left(u^{2}\right)_{yy}+hu(1-ru)\qquad t>0\,,\,0<\alpha\le 2 \qquad (11)$$

$$u(x,y,0)=e^{\left(\sqrt{\frac{hr}{8}}\right)(x+y)}$$

Using projected differential transform method of equation (11) w e have

ISSN 2350 - 0328

International Journal of Advanced Research in Science, Engineering and Technology

Vol. 1, Issue 3, October 2014

$$\frac{\Gamma\left(1+\alpha+\frac{k}{q}\right)}{\Gamma\left(1+\frac{k}{q}\right)}u(X,k+\alpha q)=\left(\sum_{m=0}^{k}u(X,m)u(X,k-m)\right)_{xx}+\left(\sum_{m=0}^{k}u(X,m)u(X,k-m)\right)_{yy}$$

$$-hr\sum_{m=0}^{k}u(X,m)u(X,k-m)+hu(X,k)\quad,\ k=0,\alpha q,2\alpha q,\dots\dots$$

$$u(X,k\alpha q)=\frac{\Gamma(1+\alpha(k-1))}{\Gamma(1+\alpha k)}\Bigg[\left(\sum_{m=0}^{\alpha q(k-1)}u(X,m)u(X,\alpha q(k-1)-m)\right)_{xx}+\left(\sum_{m=0}^{\alpha q(k-1)}u(X,m)u(X,\alpha q(k-1)-m)\right)_{yy}$$

$$-hr\sum_{m=0}^{\alpha q(k-1)}u(X,m)u(X,\alpha q(k-1)-m)+hu(X,\alpha q(k-1))\Bigg]$$

$$u(X,\alpha q)=\frac{1}{\Gamma(1+\alpha)}\left[\left(u^2(X,0)\right)_{xx}+\left(u^2(X,0)\right)_{yy}-hru^2(X,0)+hu(X,0)\right]$$

$$=\frac{h\,e^{\left(\sqrt{\frac{hr}{8}}\right)(x+y)}}{\Gamma(1+\alpha)}$$

$$u(X,2\alpha q)=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\Bigg[\left(2u(X,0)u(X,\alpha q)\right)_{xx}+\left(2u(X,0)u(X,\alpha q)\right)_{yy}$$

$$-2hru(X,0)u(X,\alpha q)+hu(X,\alpha q)\Bigg]=\frac{h^2\,e^{\left(\sqrt{\frac{hr}{8}}\right)(x+y)}}{\Gamma(1+2\alpha)}$$

$$u(X,3\alpha q)=\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\Bigg[\left(u^2(X,\alpha q)+2u(X,0)u(X,2\alpha q)\right)_{xx}+\left(u^2(X,\alpha q)+2u(X,0)u(X,2\alpha q)\right)_{yy}$$

$$-hr\left(u^2(X,\alpha q)+2u(X,0)u(X,2\alpha q)\right)+hu(X,2\alpha q)\Bigg]=\frac{h^3\,e^{\left(\sqrt{\frac{hr}{8}}\right)(x+y)}}{\Gamma(1+3\alpha)}$$

And so on in general $u(X,k\alpha q)=\dfrac{h^k\,e^{\left(\sqrt{\frac{hr}{8}}\right)(x+y)}}{\Gamma(1+k\alpha)}$

Substituting $u(X, k\alpha q)$ into equation (5) we have

$$u(X,t)=\sum_{k=0}^{\infty}\frac{h^k e^{\left(\sqrt{\frac{hr}{8}}\right)(x+y)}}{\Gamma(1+k\alpha)}t^{\alpha k}=\left(e^{\left(\sqrt{\frac{hr}{8}}\right)(x+y)}\right)\sum_{k=0}^{\infty}\frac{(ht^{\alpha})^k}{(\alpha k)!}$$

Lim $u(X,t)$ at $\alpha\to 2$ we get

$$u(X,t)=\left(e^{\left(\sqrt{\frac{hr}{8}}\right)(x+y)}\right)\sum_{k=0}^{\infty}\frac{\left(\sqrt{ht}\right)^{2k}}{(2k)!}=\left(e^{\left(\sqrt{\frac{hr}{8}}\right)(x+y)}\right)\cosh\sqrt{ht}$$

And $\alpha\to 1$ $\quad u(X,t)=\left(e^{\left(\sqrt{\frac{hr}{8}}\right)(x+y)}\right)\sum_{k=0}^{\infty}\frac{(ht)^k}{(k)!}=\left(e^{\left(\sqrt{\frac{hr}{8}}\right)(x+y)}\right)e^{ht}=e^{\left(\sqrt{\frac{hr}{8}}\right)(x+y)+ht}$

V. CONCLUSION

In this paper, we have employed a projected differential transform method (P DTM) for determining an exact solution of nonlinear time-fractional biological population model and constructing a system of two nonlinear time-fractional partial differential equations. The method is used in a direct way without using any linearization, perturbation, polynomials or restrictive assumptions in contrast to the current methods. Also, the method reliably expresses nonlinear fractional problems in a series solution form. Thus, we conclude that projected differential transform method can be considered as an efficient method for solving linear and nonlinear problems.

REFERENCES

[1] Adel Al.rabtah, redat suat Erturk, Shaher Momani: solution of fractional oscillator By using differential transform method. Comp. math. Appli 59 (2010) 1356-1362

[2] A.M.A. El-Sayed, S.Z. Rida, A.A.M. Arafa, Exact solutions of fractional-order biological population model, Commun. Theor. Phys., 52, pp. 992–996, 2009

[3] Aytac Arikoglu, Ibrahim Ozkol: Solution of fractional differential equations by using differential transform method. Chaos. Solutions . Fractals 34(2007)1473-1481

[4] Aytac Arikoglu, Ibrahim Ozkol: Solution of fractional integro – differential equations by using fractional differential transform method. Chaos. Solution. Fractals 40(2009) 251-529.

[5] F. Shakeri, M. Dehghan, Numerical solution of a biological population model using He's variational iteration method, Comput. Math. Appl., 54, pp. 1197–1209, 2007

[6] Hüseyin Koçaka, Ahmet Yıldırımb;c, An efficient new iterative method for finding exact solutions of nonlinear time-fractional partial differential equations, Nonlinear Analysis: Modelling and Control, 2011, Vol. 16, No. 4, 403–414

[7] P. Roul, Numerical solutions of time fractional degenerate parabolic equations by variational iteration method with Jumarie-modified Riemann–Liouville derivative, Math. Methods Appl. Sci., 34(9), pp. 1025–1035, 2011

[8] Salih M. Elzaki: Exact Solution of the(3+1) dimensional nonlinear Schrodinger equation using differential transform method, International Journal of Applied Mathematical Sciences ISSN 0973-0176 Volume 7, Number 1 (2014), pp. 41-46

[9] Salih M. Elzaki: Solution of Burger's Equation Using Projected Differential Transform Method, International Journal of Mathematical Sciences, ISSN:2051-5995, Vol.34, Issue.11498

[10] Salih M. Elzaki: Exact Solution of Biological Population Model Using Projected Differential Transform Method, International Journal of Mathematical and Computer Modelling, ISSN:2051- 4271, Vol.19, Issue. 1 1124

[11] Salih M. Elzaki: Solution of General System of Partial Differential Equation Using Projected Differential Transform Method, International Journal of Applied Mathematics, ISSN: 2051-5227, Vol.29, Issue.1 1271

[12]. Vedat Suat Ertrk. Shaher Momani: Solving systems of fractional differential equations using differential transform method. Journal of Compu. Appli. Math 215 (2008) 142-151.