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Convergence Analyses of Partial Differential Equation Using WAVELETS-SCHUR COMPLEMENT Method

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ABSTRACT: This paper is concerned with the analysis of wavelet based Schur complement method for numerical solution of partial differential equations. The exquisiteness of adopting the wavelet idea here lies in avoiding the somewhat difficult task of ensuring the invertiblilty of matrices and in detecting domain of sharp transitions, wavelets do this automatically for a large class of operators. The zest of new work follows from the fact that any first-scale compressed matrices can be conveniently processed before applying the next scale. The main advantage of this method is the matrix compression and faster convergence of solutions of partial differential equations. The methodology is demonstrated with the help of very simple steady state two-dimensional advection-dispersion equation. The proposed method is compared with the standard method of finite element.

KEY WORDS: Schur-complement method; Wavelets; Advection-Dispersion equation; Convergence.

I. INTRODUCTION

There are so many interesting physical systems which are characterized by the presence of localized structure or sharp transition, which might occur anywhere in the domain and advection-dispersion equation exhibits such discontinuity (shocks) [1-2]. To capture these sharp transitions in the solution, these domains would require very fine resolution. Popular methods such as finite element, so-called meshless and recently developed wavelet methods, to solve these problems efficiently, use adaptive grid techniques [3-6]. Wavelets based Schur complement techniques can also be profitably applied in solving partial differential equations useful in many applications, including simulation, animation, computer vision, etc. Though this technique is well understood, a lot of work has to be done for efficient implementation in complex domain, in particular to reduce computational time.

In this method, the finest scale finite element solution space is projected onto the scaling and wavelet spaces resulting in the decomposition of high- and low-scale components. Repetition of such a projection results in multi-scale decomposition of the fine scale solution. In the proposed wavelet projection method, the fine scale solution can be obtained by any other numerical method also. This approach is motivated by the interesting and stimulating paper by Vasilyev and Paolucci [7] who have developed an efficient approach by combining the advantages of wavelets and finite difference method. Subsequently the properties of the wavelet functions are exploited to eliminate the nodes from the smooth region where the wavelet coefficients will not exceed a preset tolerance. This wavelet-based multi-scale transformation hierarchically filters out the less significant part of the solution, and thus provides an effective framework for the selection of significant part of the solution. In this process, the 'big' coefficient matrix at the finest level will be calculated once for complete domain whereas the 'small' adaptive compressed coefficient matrix for a priory known localized zone of high gradient, which will be considerably less expensive to solve, will be used for the solution.

All linear and nonlinear systems of equations give rise to matrix computation. As the computer power upsurges and high resolution simulations are attempted, a method can reach its applicability limits quickly and hence there is a constant demand for new and fast matrix solvers. The motivation of this work follows from the observation that any one-scale compressed matrices can be used conveniently processed before applying the next scale. Proposed method

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will be designed from level-by-level wavelets, which is related to BCR work [8] exploring fully the sparsity exhibited. In this method, the regular pattern generated wavelets scales are not destroyed by the new scales as in BCR work. This fundamental but simple idea will be combined with Schur complement method. The vital assumption for this proposed method to work is invertibility of an approximate band matrix.

In practice, this method works equally well operators except the type for which we can provide proof.
Let R be a bounded operator:

$$R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \quad (1)$$

Assume that *D* is invertible. Then the first Schur complement is the operator

$$S_1(R) = A - BD^{-1}C. \quad (2)$$

Assume that *A* is invertible. Then the second Schur complement is the operator

$$S_2(R) = D - CA^{-1}B. \quad (3)$$

The reason due to which Schur complement method is useful for spectral problem is following statement [9]:
Suppose that *D* is invertible, then *R* is invertible if and only if $S_1(R)$ is invertible. Similarly, if *A* is invertible, then *R* is invertible if and only if $S_2(R)$ is invertible. The inverse is computed then by the formula

$$R^{-1} = \begin{pmatrix} S_1^{-1} & -S_1^{-1}BD^{-1} \\ -D^{-1}CS_1^{-1} & D^{-1}CS_1^{-1}BD^{-1} + D^{-1} \end{pmatrix}, \quad (4)$$

where $S_1 = S_1(R)$.

The Schur complement method also known as sub structuring method widely used in structural mechanics to solve large-scale systems. In this paper, the schur complement is used to invert a matrix and to reduce number of iterations i.e. for faster convergence. Though the I/O increases the time of inverting these matrices, this does not lessen the value of this method since it will invert matrices that are too large to reside in memory. This method hinges on blocking the matrix *R* and using the schur complement to invert the blocked matrix. Despite the existence of these high memory systems, the Schur complement methods still finds its application through parallel computing.

Matrix inversion is a major example of areas in mathematics and engineering where computers proved to be indispensable. It is difficult to take inverse of excessively large matrix. In this paper, we are going to analyze convergence behavior of wavelet schur complement method. To check accuracy of this method a comparison of this method is provided with already existing method. And to experience advantage of the method we are comparing the solution of the simple example with and without wavelets.

The rest of the paper is organized as follows. Section II gives brief introduction of the multiresolution analysis. Wavelet-splitting is discussed in section III. In section IV, Wavelet Schur complement method is discussed. Finally Section V contains numerical example of application to the new method to the solution of simple one-dimensional bar problem. Section VI includes conclusion.

II. MULTIRESOLUTION ANALYSIS

A linear span of a $\{f_j(x)\}_{j=-\infty}^{\infty}$ for $L_2(\mathbb{R})$ is dense in $L_2(\mathbb{R})$ then it is called Riesz basis and there exist positive constants $X \le Y < \infty$ (for any linear combination) such that

$$X\left\|\{c_j\}\right\|_{l^2}^2 \le \left\|\sum_{j=-\infty}^{\infty} c_j f_j(x)\right\|_2^2 \le Y\left\|\{c_j\}\right\|_{l^2}^2, \quad (5)$$

where

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$$\left\|\{c_j\}\right\|_{l^2}^2 = \sum_{j=-\infty}^{\infty}\left|c_j\right|^2. \qquad (6)$$

A function $f \in L_2(\mathbb{R})$ is called *R*-function if the sequence defined by

$$\psi_{j,k} = 2^{j/2}\psi(2^j x - k), \qquad j,k = 0,\pm 1,\pm 2\ldots \qquad (7)$$

form a Riesz basis in $L_2(\mathbb{R})$.

Direct sum of $L_2(\mathbb{R})$ is also generated by wavelets. The partial sums of this direct sum will generate a function called the scaling function, characterized by the multiresolution analysis (MRA) property of a wavelet. Recall that a wavelet function (7) is associated with its basis functions $\psi_{j,k} = 2^{j/2}\psi(2^j x - k), j,k = 0,\pm 1,\pm 2\ldots$, where the first index *j* refers to the basis resolution (dilation of ψ) while the second index *k* implements the space covering (translation of ψ).

Let W_j be the subspace formed by those precisely, the closure of linear span of wavelet function

$$W_j = clos_{L_2(\mathbb{R})}\{\psi_{j,k} \mid k = 0,\pm 1,\pm 2\ldots\ldots\}$$

Define the closed subspace (the partial sum of W_ℓ's up to $j-1$) for any j

$$V_j = \sum_{\ell=-\infty}^{j-1} W_\ell = \ldots + W_{j-2} + W_{j-1.} \qquad (8)$$

The clearly (as all basis functions $\psi_{j,k}$ are included)

$$L_2(\mathbb{R}) = \sum_{j=-\infty}^{\infty} W_j = clos_{L_2(\mathbb{R})}\left(\bigcup_{j=-\infty}^{\infty} V_j\right). \qquad (9)$$

We now consider how V_j can be generated by some *R*-function $\phi(x)$

$$V_j = clos_{L_2(\mathbb{R})}\{\phi_{j,k} \mid k = 0,\pm 1,\pm 2\ldots\ldots\} \qquad (10)$$

with $\phi_{j,k} = 2^{j/2}\phi(2^j x - k), j,k = 0,\pm 1,\pm 2\ldots$

A function $\phi \in L_2(\mathbb{R})$ is called a multiresolution analysis (MRA) and therefore a scaling function, if the sequence of subspaces V_j as from (10) satisfies

- $V_j \subset V_{j+1}$ i.e $\ldots V_{-1} \subset V_0 \subset V_1 \ldots$
- $L_2(R) = \sum_{j=-\infty}^{\infty} W_j = clos_{L_2(R)}\left(\quad_{j=-\infty}^{\infty} V_j\right).$
- $\bigcap_{k=-\infty}^{\infty} V_k = 0.$
- $f(x) \in V_k \iff f(2x) \in V_{k+1}$.
- $\{\phi_{0,k}\} = \{\phi(x-k)\}$ forms a Riesz basis for the subspaces V_0.

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Approximating a function $f \in L_2(\mathbb{R})$ by its projection $P_j f$ onto the space V_j and the projection of f on W_j as $Q_j f$ .Then we have

$$P_j f = P_{j-1} f + Q_{j-1} f. \quad (11)$$

III. WAVELET SPLITTING OF AN OPERATOR

As previously done, following the same pattern [8, 10-12], the filter coefficient c_i's and d_i's define the scaling function $\phi(x)$ and wavelet function $\psi(x)$. Further, the dilation and translation of $\phi(x)$ and $\psi(x)$ define a multiresolution analysis for L_2 in d-dimensions,

$$\begin{aligned} L_2(\mathbb{R}^d) &= \bigoplus_{k=-\infty}^{\infty} W_k = V_0 \bigoplus_{k=0}^{\infty} W_k \\ &= V_0 \oplus W_0 \oplus W_1 \ldots\ldots \oplus W_{\ell-1} \oplus W_\ell \oplus \ldots \end{aligned} \quad (12)$$

In numerical realization, we select a finite dimensional space (in finest scale) as our approximation space to the infinite decomposition of L_2 in (12), i.e. effectively

$$V_\ell = V_0 \bigoplus_{k=0}^{\infty} W_k \quad (13)$$

is used to approximate $L_2(\mathbb{R}^d)$. Consecutively, for a given operator $R : L_2 \to L_2$, its infinite and exact operator representation in wavelet bases

$$\begin{aligned} R &= P_0 R P_0 + \sum_{j=0}^{\infty}\left(P_{j+1} R P_{j+1} - P_j R P_j\right) \\ &= P_0 R P_0 + \sum_{j=0}^{\infty}\left(Q_{j+1} R Q_{j+1} + Q_j R P_j + P_j R Q_j\right) \end{aligned} \quad (14)$$

is approximated in V_ℓ by

$$R_\ell = P_\ell R P_\ell = P_0 R P_0 + \sum_{j=0}^{\ell-1}\left(Q_{j+1} R Q_{j+1} + Q_j R P_j + P_j R Q_j\right), \quad (15)$$

where $P : L_2 \to V_j$ and $Q = P_{j+1} - P_j : L_2 \to V_{j+1} - V_j = Q_j$ are both projector operators. For brevity, define operators

$$\begin{aligned} A_j &= Q_j R Q_j : W_j \to W_j, & B_j &= Q_j R P_j : V_j \to W_j, \\ C_j &= P_j R Q_j : W_j \to V_j, & D_j &= P_j R P_j : V_j \to V_j. \end{aligned} \quad (16)$$

Then one can observe that

$$R_j = A_j + B_j + C_j + D_j. \quad (17)$$

Further observation based on $R_{j+1} : V_{j+1} \to V_{j+1}$ and $V_{j+1} = V_j + W_j$ is that wavelet coefficients of R_{j+1} will be equivalently generated by block operator

$$\begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix} : \begin{pmatrix} W_j \\ V_j \end{pmatrix} \to \begin{pmatrix} W_j \\ V_j \end{pmatrix}. \quad (18)$$

Now we change the notation and consider the discretization of all continuous operators. Assume that A on the finest level (scale) $j = \ell$ is of dimension $\tau_\ell = n$. Then dimension of matrices on a coarse level j is $\tau_\ell = \tau / 2^{-j}$ for $j = 0,1,2....\ell$. The operator splitting in (18) for the case of d = 1 (higher dimensions can be discussed similarly [8,13]) corresponds to the two-dimensional wavelet transform

$$\tilde{R}_{j+1} = T_{j+1} R_{j+1} T_{j+1}^{T} = \begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix}_{\tau_{j+1} x \tau_{j+1}} \quad (19)$$

where the one level transform from $j+1$ to j (for any $j = 0,1,2\ldots.\ell$) is

$$T_{j+1} = (T_{j+1})_{\tau_{j+1} x \tau_{j+1}} = \begin{bmatrix} P_j \\ Q_j \end{bmatrix}, \quad (20)$$

$$A_j = (A_j)_{\tau_{j+1} x \tau_{j+1}} = Q_j R_{j+1} Q_j^T : \quad , \quad B_j = (B_j)_{\tau_{j+1} x \tau_{j+1}} Q_j R_{j+1} P_j^T ,$$
$$C_j = (C_j)_{\tau_{j+1} x \tau_{j+1}} = P_j R_{j+1} Q_j^T \quad , \quad D_j = (D_j)_{\tau_{j+1} x \tau_{j+1}} = P_j R_{j+1} P_j^T , \quad (21)$$

With rectangular matrices P_j and Q_j

$$P_j = \begin{bmatrix} c_0 & c_1 & \ldots. \ldots & & & c_{m-1} \\ & c_0 & c_1 & \ldots & & c_{m-1} \\ & & . & . & . & \\ & & & . & . & . \\ & & & . & . & . \\ c_2 & c_3 & \ldots\ldots c_{m-1} & & c_0 & c_1 \end{bmatrix}_{\tau_j x \tau_{j-1}} , \quad (22)$$

$$Q_j = \begin{bmatrix} d_0 & d_1 & \ldots. \ldots & & & d_{m-1} \\ & d_0 & d_1 & \ldots & & d_{m-1} \\ & & . & . & . & \\ & & & . & . & . \\ & & & . & . & . \\ d_2 & d_3 & \ldots\ldots d_{m-1} & & d_0 & d_1 \end{bmatrix}_{\tau_j x \tau_{j-1}} . \quad (23)$$

To observe a relationship between the above level-by-level form and the standard wavelet representation, define a square matrix of size $\tau_\ell \times \tau_\ell = n \times n$ for any $j = 0,1,2\ldots.\ell$

$$\overline{T} = \begin{bmatrix} I_{\nu_j} & \\ & T_j \end{bmatrix}, \quad (24)$$

where $\nu_j = n - \tau_j$; clearly; $\nu_j = 0$ and $\overline{T}_\ell = T_\ell$ Then the standard wavelet transform can be written as:

$$T = \overline{T}_1 \ldots.. \overline{T}_{\ell-1} \overline{T}_\ell , \quad (25)$$

that transforms matrix R into $\tilde{R} = TRT^T$.

Thus the diagonal blocks of $\tilde{A}$ are same as that of A_j of a level-by-level form. However off-diagonal blocks of the former are different from of the B_j and C_j of the latter. To gain some insight of the off-diagonal blocks of matrix $\tilde{A}$ with standard wavelet transform.

$$\tilde{R}_3 = \overline{T}_3 R \overline{T}_3^T = T_3 R_3 T_3^T = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}_{nxn} \quad (26)$$

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$$\tilde{R}_2 = \bar{T}_2 \tilde{R}_3 \bar{T}_2^T = \begin{bmatrix} A_2 & B_2 T_2^T \\ T_2 C_2 & \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \end{bmatrix}_{nxn} \quad (27)$$

$$\tilde{R}_1 = \bar{T}_1 \tilde{R}_2 \bar{T}_1^T = \begin{bmatrix} A_2 & B_2 T_2^T \begin{bmatrix} I_{3n/4} & \\ & T_1^T \end{bmatrix} \\ \begin{bmatrix} I_{3n/4} & \\ & T_1 \end{bmatrix} T_2 C_2 & \begin{bmatrix} A_1 & B_1 T_1^T \\ T_1 C_1 & \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} \end{bmatrix} \end{bmatrix}_{nxn}. \quad (28)$$

Clearly, the off-diagonal blocks of $\tilde{R}_1$ are perturbation of that of the level-by-level form off-diagonal blocks B_j and C_j; in fact the one sided transforms of the off-diagonal blocks are responsible for the resulting sparsity structure.

IV. SCHUR COMPLEMENT METHOD WITH THE LEVEL-BY-LEVEL WAVELETS

Wavelet Schur complement method for solving linear system $Rx = b$ defined on the finest scale V_ℓ i.e.

$$R_\ell x_\ell = b_\ell \quad (29)$$

where $R_\ell = R$ is of size $\tau_\ell x \tau_\ell = n x n$ as discussed previously $x_\ell, b_\ell \in \mathbb{R}^n$.

Firstly, at level ℓ we consider $V_\ell = V_{\ell-1} \oplus W_{\ell-1}$ and the wavelet transform of (29) yields

$$\tilde{R}_\ell \tilde{x}_\ell = \tilde{b}_\ell, \quad (30)$$

Where $\tilde{x}_\ell = T_\ell x_\ell$ and $\tilde{b}_\ell = T_\ell b_\ell$. Since

$$\tilde{R}_\ell = \begin{bmatrix} A_{\ell-1} & B_{\ell-1} \\ C_{\ell-1} & D_{\ell-1} \end{bmatrix}_{nxn}. \quad (31)$$

In the decomposed form,

$$\begin{bmatrix} \bar{A}_{\ell-1} & \bar{B}_{\ell-1} \\ \bar{C}_{\ell-1} & \bar{D}_{\ell-1} \end{bmatrix} \begin{pmatrix} x_\ell^{(1)} \\ x_\ell^{(2)} \end{pmatrix} = \begin{pmatrix} r_\ell^{(1)} \\ r_\ell^{(2)} \end{pmatrix}. \quad (32)$$

Using Schur complement method

$$\begin{cases} \bar{A}_{\ell-1} y_{\ell-1} = r_\ell^{(1)} \\ x_2 = r_\ell^{(2)} - \bar{C}_{\ell-1} y_{\ell-1} \\ \left(T_{\ell-1} - \bar{C}_{\ell-1} \bar{A}^{-1} \bar{B}_{\ell-1}\right) x_\ell^{(2)} = y_2 \\ x_\ell^{(1)} = y_{\ell-1} - \bar{A}^{-1} \bar{B}_{\ell-1} x_\ell^{(2)} \end{cases} \quad (33)$$

So final solution we obtain $[x_1; x_2]$.

V. RESULTS AND DISCUSSIONS

Matrix compression and faster convergence are some reasons due to which wavelets expansions works with great success. In this paper, wavelet-Schur complement method is used for matrix compression and faster convergence. In this new method, the stiffness matrix possesses the desirable properties suitable for using Schur complement method. The partial differential equations are discretized in the space domain using linear elements, wavelet based finite

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element method is used for analyses. In order to have a check accuracy and stability of the method, a comparative study is been carried out for one dimensional bar problem.

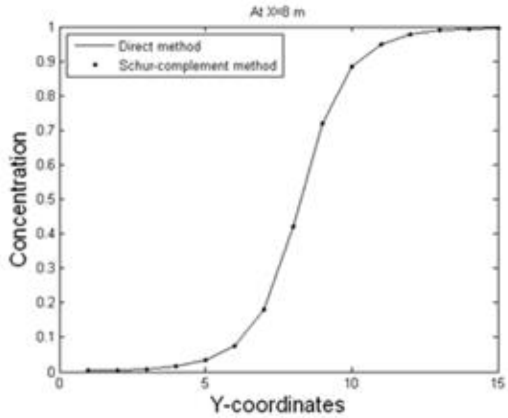

Fig. 1. Comparison of solution of 2d steady state advection-dispersion equations obtains by Wavelet Schur complement method and direct method.

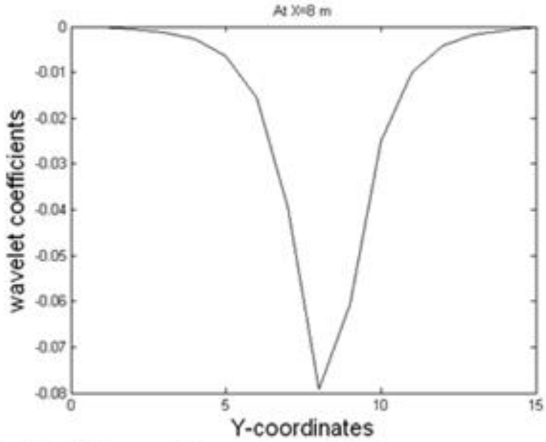

Fig. 2. Corresponding wavelets coefficients.

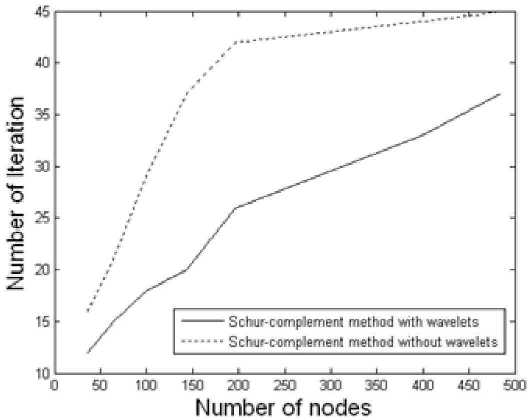

Fig. 3. Convergence behavior of Schur complements method with wavelets and without wavelets and direct method.

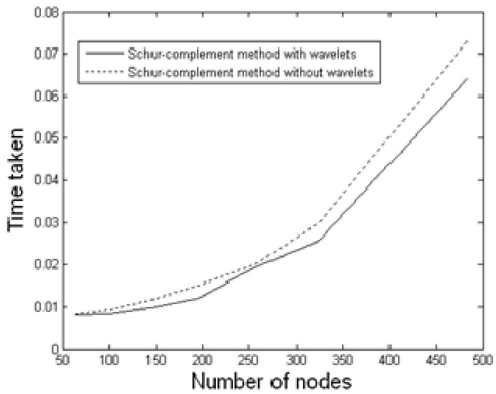

Fig. 4. Comparison of computational time taken by Schur complement method with wavelets and without wavelets.

The results for two dimensional steady state advection-dispersion equations are compared with standard finite element method and this is depicted in Figs. 1. It can be noted form the figure that there is good agreement among the results. In Fig. 2, corresponding wavelets coefficients is depicted, it can be observed that changes in wavelet coefficients are very small compared to the scaling coefficients and start well before the sudden jump. This helps in automatic grid refinement at the location of sudden jump. It can be easily observed from Fig. 3, that number of iteration taken by Schur complement method with wavelets is smaller than that without wavelets, which shows that wavelets Schur complement method converges faster than Schur complement method without wavelets. The computational time taken by Schur complement method with wavelets are compared with the results obtained by using Schur complement method without wavelets, and are shown in Fig. 4. It can be noted form the figures that time taken by Schur complement method with wavelets is lesser than that of Schur complement method without wavelets which shows that Wavelet Schur complement method is more efficient. The solution of two dimensional steady state advection-dispersion equations establishes the efficacy of the proposed technique. The advantage of the method is not clearly visible due to simplicity of the problem taken. The methodology developed here can be extended to higher dimensional problems.

VI. CONCLUSION

The Wavelets Schur complement method presented here provides both speedup and memory effectiveness which can be qualified to the reduction in matrix sizes while taking inverse and number of iteration. Using this algorithm large-

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scale problems, which could not be solved in earlier studies, can now be solved. The proposed results give better results for larger systems and use of many sub domains (processor) do not decline the speedup. The choice of dividing a domain into subdomain is crucial. It directly affects the number of unknowns on the interface, computational loading of subdomains and hence the overall efficiency. The platform properties directly affect the efficiency of the algorithm. Therefore making a general statement about the efficiency of the algorithm by testing it on a single platform might lead unrealistic results. To avoid this, properties of the platform and algorithm must be studied carefully.

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