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Stability Theorems for Stochastic Differential Equations (S.D.E.'s) with Memory (Part 1)

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ABSTRACT:Here stochastic differential equations with memory means delay stochastic differential equations. In the present work we have formulated an example of the main delay stochastic differential equation,see [2] and [11] and [9]. The example which we have considered is of the following form:

$$d\begin{pmatrix} x_1(t) \\ x_2(t) \\ \tilde{x}^t(t) \end{pmatrix} = \begin{pmatrix} x_2(t) \\ -x_1(t) + \int_0^\infty e^{-s}(x^t{}_1(s) - x_1(t))ds \\ -(\tilde{x}^t)' - x_2(t) \end{pmatrix} dt + \begin{pmatrix} 0 \\ \alpha\|\tilde{x}^t\| \\ 0 \end{pmatrix} dW_t$$

where the ordered triple $(x_1(t), x_2(t), \tilde{x}^t)$ can be considered as representing position, velocity and history ofposition respectively. We will call the space containing this triple "the history space X". In section two of thiswork we have proved a stability theorem for a diffusion of a S.D.E. in R^n. With a suitable choice of Lyapunovfunctional we have proved that the motion will finally come to rest at the origin. In section three we haveextended the space R^n of section two to a history space, i.e. to a space with three components; position,velocity and history of position. Also we formulate our S.D.E. on this history space X and also we found thegenerator of the diffusion. The initiation of the present work was suggested by Prof. Maassen,J.D.M.,KatholikUniversity of Nijmegen,The Netherlands.

I. INTRODUCTION

Stochastic Differential Equations with memory serve as models of noisy physical processes whose time evolution depends on their past history. In physics, lazer dynamics with delayed feedback is often investigated as well as the dynamics of noisy bistable systems with delay. In biophysics, stochastic equations are used to model delayed visual feedback systems or human postural way. For more details see the website of Prof. Salah-E.A.Mohammed namely “sfde.math.siu.edu’’. By “with memory” we mean a S.D.E. in which the initial process is defined on an interval of time in the past and not at a particular point as in the ordinary S.D.E.’s.

II. RELATED WORKS

In [4] Delfour and Mitter studied the existence and uniqueness of solutions of ordinary differential equationswith constant delays and discontinuous initial data. In [9] Mohammed proved the existence and uniquenessof solutions for stochastic functional differential equations (S.F.D.E.’S) with continuous initial data. Since forpractical reasons the initial data may not be continuous, we have generalized the work of Mohammed to the casein which the initial data are not necessarily continuous using the Hilbertian norm on $R^n \times L^2([-1,0], R^n)$ $(n \in N)$ instead of the supremum norm on $C([-1,0], R^n)$. The first norm has advantages over the second one whendealing with the Markov property. Moreover we establish a stronger version of the existence and uniquenesstheorem so obtained, See the M.Sc [2]. See also chapter One in the Ph.D Thesis of Ahmed [1] and also on page 226 of [9].The website of Mohammed namely “sfd.math.siu.edu”.The work in this paper can be consideredas a generalization of the work of Mizel and Trutzer in [11].

ISSN: 2350-0328

International Journal of Advanced Research in Science, Engineering and Technology

Vol. 1, Issue 4 , November 2014

III. A STABILITY THEOREM IN R^n

Here we shall prove some stability theorems for the solution process of a S.D.E. of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t \qquad (III.1)$$

where $b: R^n \to R^n$ and $: R^n \to R$ $(n \geq 1)$ are given by $b(x) = -x$ and $\sigma(x) = \alpha\|x\|, \alpha^2 < \frac{2}{n}$and $W_t$is a normalized n-dimensional Brownian motion based on some probability spaceΩ, and X_t is an R^n valuedstochastic process.

Let $V: R^n \to R$ be a Lyapunov functional given by $V(x) = \frac{1}{2}\|x\|^2$. Let Q be an open subset of R^n.Let τ bethe exit time from Q: $\tau = inf\{t: x_t \notin Q\}$and $\tau = \infty$ if the motion does not hit the boundary of Q forever.

Put $\tau(t) = \tau \wedge t$. Now as $b(x) = -x$ and $\sigma(x) = \alpha\|x\|$satisfy

$$\|b(x) - b(y)\| = \|-x + y\| = \|x - y\|$$

And

$$\begin{aligned}\|\sigma(x) - \sigma(y)\| &= |\alpha\|x\| - \alpha\|y\|| \\ &= \alpha|\|x\| - \|y\|| \\ &\leq \alpha\|x - y\|\end{aligned}$$

then$b$ and σ satisfy the Lipschitz condition. Also it is easy to see that b and σ satisfy the linear growthcondition. Thus the considered S.D.E. has a strongly unique solution x_t. Then x_t is a continuous strongMarkov process with weak infinitesimal generator A and the operator A is given by

$$(Af)(x) = -x \cdot \nabla f(x) + \frac{1}{2}\alpha^2\|x\|^2 \cdot \Delta f(x),$$

where$f$is twice continuously differentiable and bounded on Q.

In order to find $AV(x)$, let us calculate:

$$\begin{aligned}\Delta V(x) = \nabla \frac{1}{2}\|x\|^2 &= \frac{1}{2}\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right)({x_1}^2 + {x_2}^2 + \dots + {x_n}^2) \\ &= \frac{1}{2}(2x_1, 2x_2, \dots, 2x_n) = (x_1, x_2, \dots, x_n) = x \quad ;\end{aligned}$$

$$\begin{aligned}\Delta V(x) &= \left(\frac{\partial^2}{\partial x_1} + \frac{\partial^2}{\partial x_2} + \dots + \frac{\partial^2}{\partial x_n}\right) \cdot \frac{1}{2}({x_1}^2 + {x_2}^2 + \dots + {x_n}^2) \\ &= \frac{1}{2}(2 + 2 + \dots + 2) \\ &= (1 + 1 + \dots + 1) = n.\end{aligned}$$

Thus

$$AV(x) = -\langle x, x\rangle + \frac{1}{2}\alpha^2\|x\|^2 n = -\|x\|^2 + \frac{n}{2}\alpha^2\|x\|^2 = \|x\|^2\left(\frac{n\alpha^2}{2} - 1\right) \leq 0,$$

and hence $AV(x) \leq 0 \quad \forall x \in Q$(as $\alpha^2 < \frac{2}{n}$).

III.1 Theorem: Let $\tau(t) = \tau \wedge t$ and $x_{\tau(t)} = x_{\tau\wedge t}$. Employing the above assumptions and setting we have thefollowing for $x = x_0 \in Q$

i. $V(x_{\tau\wedge t})$is a non-negative supermartingale.

ii. $P_x\{sup_{0\leq t V(x_{\tau\wedge t})} \geq q\} \leq \frac{V(x)}{q} \quad \forall q > 0$.

iii. There exists a random variable $v \geq 0$ such that $V(x_{\tau(t)}) \to v$with probability 1.

Proof:

i. Let$x \in Q$,thenas$V$istwicecontinuouslydifferentiableandboundedon$Q$, and also$\tau(t) = \tau \wedge t$isboundedandhence$E\tau(t) < \infty$, thuswecanuseDynkin'sformulatoget

$$E_x V(x_{\tau(t)}) = V(x) + E_x \int_0^{\tau(t)} AV(x_s)ds$$

$$E_x V(x_{\tau(t)}) - V(x) = E_x \int_0^{\tau(t)} AV(x_s)ds \leq 0$$

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International Journal of Advanced Research in Science, Engineering and Technology

Vol. 1, Issue 4 , November 2014

because$AV(x_s) \le 0 \quad \forall s \le \tau(t)$ as proved.Hence $E_x V\left(x_{\tau(t)}\right) \le V(x)\, \forall x \in Q$, i.e. $V(x_t)$ is a non-negative supermartingale.

ii. This is just the martingale inequality.

iii. As $V(x_{\tau\wedge t})$ is a non-negative supermartingale, it converges a.s. to a non-negative random variable, say$v$.

We shall now show that $v = 0$ i.e. $V(x_t) \to 0$as $t \to \infty$. To do this we shall find an expression for$V(X_t) = \frac{1}{2}\|X\|^2$ in terms of t and then take the limit as $t \to \infty$.Morespecifically, we shall show that$EV(X_t) = E\frac{1}{2}\|X_t\|^2 = \frac{1}{2}E\|X_0\|^2 e^{(n\alpha^2-2)t}$where $X_0 = x \in Q$and $x \ne 0$. Now since we have $\alpha^2 < \frac{2}{n}$, i.e. $(2 - n\alpha^2) > 0$, thus

$$\lim_{t\to\infty} e^{(n\alpha^2-2)t} = \lim_{t\to\infty}\left(\frac{1}{e^{(2-n\alpha^2)t}}\right) = 0$$

$$\begin{aligned}\lim_{t\to\infty} EV(X_t) &= \lim_{t\to\infty}\frac{1}{2}E\|X_0\|^2 e^{(n\alpha^2-2)t}\\ &= \frac{1}{2}E\|X_0\|^2 \lim_{t\to\infty} e^{(n\alpha^2-2)t}\\ &= \frac{1}{2}E\|X_0\|^2 \cdot 0 = 0\end{aligned}$$

Now in order to prove that $V(x_t) \to 0$ as $t \to \infty$ it remains to prove the following claim.

Claim:

$$E\|X_t\|^2 = E\|X_0\|^2 e^{(n\alpha^2-2)t}.$$

Proof of Claim.Rewrite the S.D.E. (III.1) in the form

$$dX_t^i = -X_t^i dt + \alpha X_t^i dW_t \qquad (i = 1, 2, \ldots, n)$$

□

Where $X_t = (X_t^1, X_t^2, \ldots, X_t^n)$and $W_t = (W_t^1, W_t^2, \ldots, W_t^n)$, where the W_t^i's are n independent copies of the Brownian motion.

To find $d\|X_t\|^2$, let us calculate

$$\langle dW_t, dW_t\rangle = \sum_{i=1}^{n}\left(dW_t^i\right)^2 = \sum_{i=1}^{n} dt = ndt. \qquad (III.2)$$

Now by using the Itômultiplication table

	dt	dW_t
dt	0	0
dW_t	0	dt

We defined that

$$\begin{aligned}\langle dX_t, dX_t\rangle &= \langle \alpha\|X_t\|dW_t, \alpha\|X_t\|dW_t\rangle\\ &= \alpha^2\|X_t\|^2\langle dW_t, dW_t\rangle\\ &= \alpha^2\|X_t\|^2 ndt \text{by}(III.2)\end{aligned}$$

Now by using Itô formula we have

$$\begin{aligned}d\|X_t\|^2 &= 2\langle X_t, dX_t\rangle + \langle dX_t, dX_t\rangle\\ &= 2\langle X_t, -X_t dt + \alpha\|X_t\|dW_t\rangle + n\alpha^2\|X_t\|^2 dt\\ &= 2\langle X_t, -X_t dt\rangle + 2\langle X_t, \alpha\|X_t\|dW_t\rangle + n\alpha^2\|X_t\|^2 dt\\ &= -2\|X_t\|^2 dt + n\alpha^2\|X_t\|^2 dt + 2\alpha\|X_t\|X_t dW_t\\ &= (n\alpha^2 - 2)\|X_t\|^2 dt + 2\alpha\|X_t\|X_t dW_t\end{aligned}$$

Now

$$\begin{aligned}dE\|X_t\|^2 &= (n\alpha^2 - 2)E\|X_t\|^2 dt + E2\alpha\|X_t\|X_t EdW_t\\ &= (n\alpha^2 - 2)E\|X_t\|^2 dt\end{aligned}$$

(as $EdW_t = 0$). Thus $dE\|X_t\|^2 = (n\alpha^2 - 2)E\|X_t\|^2 dt$ i.e. $\frac{d}{dt}(E\|X_t\|^2) = (n\alpha^2 - 2)E\|X_t\|^2$ which is an ordinary differential equation of the form

$$\frac{dy}{dt} = (n\alpha^2 - 2)y, \qquad y = E\|X_t\|^2$$

$$\frac{dy}{y} = (n\alpha^2 - 2)dt$$

$$\ln y = (n\alpha^2 - 2) + c$$

$$y = e^{(n\alpha^2-2)t} \cdot e^c.$$

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International Journal of Advanced Research in Science, Engineering and Technology

Vol. 1, Issue 4 , November 2014

Hence $y = c_1 e^{(n\alpha^2-2)t}$, and at $t = 0, y = y_0 = E\|X_0\|^2$. Hence $E\|X_t\|^2 = E\|X_0\|^2 e^{(n\alpha^2-2)t}$.

IV. HISTORYSPACE,S.D.E.INHISTORYSPACEANDGENERATOR INHISTORY SPACE

Let $x: R \times \Omega \to R$ be a real valued stochastic process defined on a probability space . Then the history of x up to time t is given by the map $x^t: R^+ \times \Omega \to R$where $x^t(s,\omega) = x(t-s,\omega)$ $(s \geq 0)$. For simplicity we shall omit ω and write $x^t(s) = x(t-s)$. Let $\tilde{x}: R^+ \to R$ be a process such that $\tilde{x}$ belongs to $L^2(R^+, R, e^{-s}ds)$, i.e.$\int_0^\infty \tilde{x}^2(s)e^{-s}ds < \infty$.

Let $X = R \times R \times L^2(R^+, R, e^{-s}ds)$. For $x = (x_1, x_2, \tilde{x}) \in X$ we define $\|x\|_X$by $\|x\|_X = \left(x_1^2 + x_2^2 + \int_0^\infty \tilde{x}^2(s)e^{-s}ds\right)^{\frac{1}{2}}$.

The above space X is the history space. We shall introduce a S.D.E. on the above history space X. The S.D.E.which we shall introduce is a model describing the motion of a dangling spider (see [5]). Thus let us thinkof the two real valued processes$x_1(t)$ and $x_2(t)$ as representing the position and the velocity (respectively) of the dangling spider at time t, and think of $\tilde{x}$ as representing the history of the position up to time t where$\tilde{x} \in L^2(R^+, R, e^{-s}ds)$.

Let $\sigma: X \to R$ be the real valued function defined on X such that $\sigma(x) = \alpha\|\tilde{x}\|$ where $x = (x_1, x_2, \tilde{x}) \in X$and < 1. Let W_t be a normalized one-dimensional Brownian motion. Now we are ready to formulate the S.D.E. wepromised on the history space X, as follows:

$$d\begin{pmatrix} x_1 \\ x_2 \\ \tilde{x}^t \end{pmatrix} = \begin{pmatrix} x_2(t) \\ -x_1(t) + \int\limits_0^\infty e^{-s}\int\limits_0^t (x_1^t(s) - x_1(t))ds \\ (\tilde{x}^u)' - x_2(t) \end{pmatrix} dt + \begin{pmatrix} 0 \\ \alpha\|\tilde{x}^u\| \\ 0 \end{pmatrix} dW_t \qquad (IV.1)$$

where the function $\tilde{x}^t: R^+ \to R$ is defined by $\tilde{x}^t(s) = x_1^t(s) - x_1(t) = x_1(t-s) - x_1(t)$. In integrated form equation $(IV.1)$ reads as follows:

$$\begin{pmatrix} x_1(t) - x_1(0) \\ x_2(t) - x_2(0) \\ \tilde{x}^t(s) - \tilde{x}^0(s) \end{pmatrix} = \begin{pmatrix} \int_0^t x_2(u)du \\ -\int_0^t x_1(u)du + \int_0^\infty e^{-s}\int_0^t \tilde{x}^u(s)duds \\ -\int_0^t (\tilde{x}^u)'(s)du - \int_0^t x_2(u)du \end{pmatrix} + \begin{pmatrix} 0 \\ \int_0^t \alpha\|\tilde{x}^u\| \\ 0 \end{pmatrix} dW_u. \qquad (IV.2)$$

Observe that the third row and the first row of the S.D.E. $(IV.1)$ are equivalent. This can be seen by observing that the first row leads to equation $(IV.3)$ and the third row leads to equation $(IV.4)$, and by comparing the left hand sides and the right hand sides of the following two equations $(IV.3)$ and $(IV.4)$:

$$x_1(t) - x_1(0) = \int\limits_0^t x_2(u)du. \qquad (IV.3)$$

We also have:

$$x_1(t-s) - x_1(t) - x_1(-s) + x_1(0) = -\int\limits_0^t \frac{\partial}{\partial s}\big(x_1(u-s) - x_1(u)\big)du - \int\limits_0^t x_2(u)du$$

$$= \int\limits_0^t x_1{}'(u-s)du - \int\limits_0^t x_2(u)du$$

$$x_1(t-s) - x_1(t) - x_1(-s) + x_1(0) = x_1(t-s) - x_1(-s) - \int\limits_0^t x_2(u)du \qquad (IV.4)$$

As we have done in section 2 we can also check that the above S.D.E. has a unique solution by checking theLipschitz condition for $b(x)$ and $\sigma(x)$ where

$$b(x) = \begin{pmatrix} x_2 \\ -x_1 + \int_0^\infty e^{-s}\tilde{x}(s)ds \\ -(\tilde{x})' - x_2 \end{pmatrix} \text{and } \sigma(x) = \begin{pmatrix} 0 \\ \alpha\|\tilde{x}\| \\ 0 \end{pmatrix},$$

Where $x = (x_1, x_2, \tilde{x})$. Again the S.D.E. $(IV.2)$ can be written as

$$X_t = X_0 + \int\limits_0^t b(X_u)du + \int\limits_0^t \sigma(X_u)dW_u \qquad (IV.5)$$

Where

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Vol. 1, Issue 4 , November 2014

$$X_t = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \tilde{x}^t \end{pmatrix}$$

Now let $x, y \in X$ where $x = (x_1, x_2, \tilde{x})$and $y = (y_1, y_2, \tilde{y})$, then

$$\|b(x) - b(y)\|_X^2 \le 2|x_1 - y_1|^2 + 3|x_2 - y_2|^2 + 2\left|\int_0^\infty e^{-s}\big(\tilde{x}(s) - \tilde{y}(s)\big)ds\right|^2 + 2\int_0^\infty e^{-s}\left(-(\tilde{x})'(s) + (\tilde{y})'(s)\right)^2 ds.$$

But as $\int_0^\infty e^{-s} ds = 1$ then by Hölder inequality we have

$$\left|\int_0^\infty e^{-s}\big(\tilde{x}(s) - \tilde{y}(s)\big)ds\right|^2 \le \left(\int_0^\infty 1 \cdot e^{-s} ds\right)\left(\int_0^\infty e^{-s}\big(\tilde{x}(s) - \tilde{y}(s)\big)^2 ds\right)$$

$$\le \int_0^\infty e^{-s}\big(\tilde{x}(s) - \tilde{y}(s)\big)^2 ds$$

And

$$\int_0^\infty e^{-s}\left(-(\tilde{x})'(s) + (\tilde{y})'(s)\right)^2 ds = \int_0^\infty e^{-s}\left[\big(-\tilde{x}(s) + \tilde{y}(s)\big)'\right]^2 ds$$

$$= \int_0^\infty e^{-s}\left[-\big(-\tilde{x}(s) + \tilde{y}(s)\big)\right]^2 ds$$

$$= \int_0^\infty e^{-s}\big(\tilde{x}(s) - \tilde{y}(s)\big)^2 ds$$

Hence

$$\|b(x) - b(y)\|_X^2 \le 2|x_1 - y_1|^2 + 3|x_2 - y_2|^2 + 4\int_0^\infty e^{-s}\big(\tilde{x}(s) - \tilde{y}(s)\big)^2 ds$$

$$\le 4\left[|x_1 - y_1|^2 + |x_1 - y_1|^2 + \int_0^\infty e^{-s}\big(\tilde{x}(s) - \tilde{y}(s)\big)^2 ds\right]$$

$$\le \|x - y\|_X^2.$$

Also it is clear that $\sigma(x)$ is also Lipschitz and the linear growth condition is automatically satisfied. Thus $b$and σ satisfy the following inequalities:

$$\|b(x)\|_X + \|\sigma(x)\|_X \le K_1(\|x\|_X + 1) \qquad (IV.6)$$

and

$$\|b(x) - b(y)\|_X + \|\sigma(x) - \sigma(y)\|_X \le K_2(\|x - y\|_X) \qquad (IV.7)$$

where$x, y \in X$ and K_1 and K_2 are constants. Hence the considered S.D.E. (IV.1) has a unique strong solutionX_t. Thus X_t is a continuous strong Markov process.

Now we shall find the generator A for the above diffusion $X_t$restricting attention to the Lyapunov function$V: X \to R$where $V(\bar{x}) = \frac{1}{2}\|\bar{x}\|_X^2$. We shall use the definition of the generator in the form $AV(x) = \frac{d}{dt}EV(X_t^x)|_{t=0}$.

Now

$$V(X_t) = \frac{1}{2}\|X_t\|_X^2$$

$$= \frac{1}{2}\left(x_1(t)^2 + x_2(t)^2 + \int_0^\infty e^{-s}\tilde{x}(s)^2 ds\right) \qquad (IV.8)$$

$$= \frac{1}{2}\left(x_1(t)^2 + x_2(t)^2 + \int_0^\infty e^{-s}(x_1^t(s) - x_1(t))^2 ds\right).$$

ISSN: 2350-0328

International Journal of Advanced Research in Science, Engineering and Technology

Vol. 1, Issue 4 , November 2014

Let $Q=\{x:V(x)<q\}$ i.e. $Q=\left\{x:\frac{1}{2}\|x\|_X^2<q\right\}=\{x:\|x\|_X<\sqrt{2q}\}$. Thus Q is bounded subset of X. Now let τ be the exit time from Q, i.e. $\tau=inf\{t:x_t\notin Q\}$ and let $\tau=\infty$ if the motion does not hit the boundary of Q forever.

Suppose that the motion starts at point $X_0=x\in Q$. We wish to calculate $dV(X_t^x)=d\frac{1}{2}\|X_t\|_X^2$. We have by Itô'sformula

$$d\|X_t\|_X^2=2\langle X_t,dX_t\rangle+\langle dX_t,dX_t\rangle. \qquad (IV.9)$$

Now

$$\langle X_t,dX_t\rangle=\begin{pmatrix}x_1(t)\\x_2(t)\\\tilde{x}^t\end{pmatrix}\cdot\begin{pmatrix}x_2(t)dt\\(-x_1(t)+\int_0^\infty e^{-s}\tilde{x}^t(s)ds)dt+\alpha\|\tilde{x}\|\,dW_t\\-(\tilde{x}^t)'dt-x_2(t)dt\end{pmatrix}$$

$$=x_1(t)x_2(t)dt-x_1(t)x_2(t)dt+x_2(t)\int_0^\infty e^{-s}\tilde{x}^t(s)dsdt-\langle\tilde{x}^t,(\tilde{x}^t)'\rangle dt+\alpha\|\tilde{x}\|x_2(t)dW_t$$

$$-\left(x_2(t)\int_0^\infty e^{-s}\tilde{x}^t(s)ds\right)dt$$

$$=-\langle\tilde{x}^t,(\tilde{x}^t)'\rangle dt+\alpha\|\tilde{x}\|x_2(t)dW_t.$$

Now denoting $\tilde{x}^t$ by $f$we have:

$$\langle\tilde{x}^t,(\tilde{x}^t)'\rangle=\langle f,f'\rangle=\int_0^\infty f(s)'f(s)e^{-s}ds$$

$$=\int_0^\infty e^{-s}d\left(\frac{1}{2}f^2(s)\right)$$

$$=e^{-s}\frac{1}{2}f^2(s)\Big|_0^\infty+\frac{1}{2}\int_0^\infty e^{-s}f^2(s)$$

$$=\frac{1}{2}\int_0^\infty e^{-s}f^2(s)=\frac{1}{2}\int_0^\infty e^{-s}\tilde{x}^2(s)$$

$$=\frac{1}{2}\int_0^\infty e^{-s}(x^t(s)-x_1(t))^2ds.$$

Note $e^{-s}\frac{1}{2}f^2(s)\Big|_0^\infty=0$ because $e^{-\infty}=0$ and $\tilde{x}(\infty)<\infty$ and $f(0)=\tilde{x}^t(0)=x_1^t(0)-x_1(t)=x_1(t)-x_1(t)=0.$

Hence

$$\langle X_t,dX_t\rangle=-\frac{1}{2}\int_0^\infty e^{-s}(\tilde{x}^t(s))^2dsdt+\alpha\|\tilde{x}^t\|x_2(t)dW_t \qquad IV.10$$

Now by using Itômultiplication table we have

$$\langle dX_t,dX_t\rangle=\alpha^2\|\tilde{x}^t\|^2dt. \qquad (IV.11)$$

Thus by equations $(IV.9)$, $(IV.10)$ and $(IV.11)$ we get

$$d\|X_t\|_X^t=-\int_0^\infty e^{-s}(\tilde{x}^t(s))^2dsdt+2\alpha\|\tilde{x}^t\|x_2(t)dW_t+\alpha^2\|\tilde{x}^t\|^2dt. \qquad (IV.12)$$

Now by $(IV.8)$ we have

$$dV(X_t)=d\frac{1}{2}\|X_t\|_X^t$$

$$=-\frac{1}{2}\int_0^\infty e^{-s}(\tilde{x}^t(s))^2dsdt+\frac{1}{2}\alpha^2\|\tilde{x}^t\|^2dt+\alpha\|\tilde{x}^t\|x_2(t)dW_t. \qquad (IV.13)$$

Now by taking expectation on both sides of $(IV.13)$ and using the fact that

$$E\alpha\|\tilde{x}^t\|x_2(t)dW_t=E\alpha\|\tilde{x}^t\|x_2(t)EdW_t$$

we get the generator of the diffusion for $V(X_t)$ as follows:

$$AV(X_t) = \int_0^\infty e^{-s}\tilde{x}^2(s)ds + \frac{1}{2}\alpha^2\|\tilde{x}\|^2$$

$$= \int_0^\infty e^{-s}\tilde{x}^2(s)ds + \frac{1}{2}\alpha^2\int_0^\infty e^{-s}\tilde{x}^2(s)ds$$

$$= \frac{1}{2}(\alpha^2 - 1)\int_0^\infty e^{-s}\tilde{x}^2(s)ds \qquad (IV.14)$$

$$= \frac{1}{2}(\alpha^2 - 1)\|\tilde{x}^t\|^2.$$

A. REMARKS

a) All the results which we have established in this work can be extended by replacing the Brownian motion W by another process $Z: [0, a] \times \Omega \to R$which is a continuous martingale adaptedto$\{\mathcal{F}_t\}_{t\in[0,a]}$ and has independent increments and satisfies with some constant $K$the inequalities

$$|E[Z(t) - Z(s)]|\mathcal{F}_s| \le K(t-s) \text{and} E\left(|Z(t) - Z(s)|^2\middle|\mathcal{F}_s|\right) \le K(t-s) \text{ for } 0 \le s \le t \le a.$$

Observe that the above properties of Z which we have just mentioned are the only properties of Wwhich we have used (in case of Brownian motion) to prove the results which we have obtained in this work.See [2].

b) All the results which we have established in this work, can be extended to a processes $f', g': [0, a] \times R^n \times L^2(J, R^n) \to L(R^m, R^n)$ $(m, n \in N)$ instead of the processes $f, g: [0, a] \times R^n \times L^2(J, R^n) \to R^n$(n ∈ N), and instead of the Brownian motion Wwe use the process $Z: [0, a] \times \Omega \to R^m$ which is a martingale adapted to$\{\mathcal{F}_t\}_{t\in[0,a]}$, continuous on $[0, a]$, and has independent increments and satisfies for some constant K the inequalities

$$|E[Z(t) - Z(s)]|\mathcal{F}_s| \le K(t-s) \text{and} E\left(|Z(t) - Z(s)|^2\middle|\mathcal{F}_s|\right) \le K(t-s) \text{ for } 0 \le s \le t \le a. \text{ See [2].}$$

c) All the lemmas and theorems in this work hold for any delay interval $J' = [-r, 0)$ $(r \ge 0)$.See [2].

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