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Observation on the Non-Homogeneous Ternary Biquadratic Equation $x^2-2xy+3y^2=(k^2+2s^2)z^4$

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ABSTRACT: We obtain infinitely many non-zero integer triples (x, y, z) satisfying the Biquadratic equation with three unknowns $x^2-2xy+3y^2=(k^2+2s^2)z^4$.Various interesting relations between the solutions and special numbers, namely, polygonal numbers, Pyramidal numbers,Star numbers, Stella Octangular numbers, Octahedral numbers, Four Dimensional Figurative numbers, Centred polygonal and pyramidal numbers are exhibited

KEYWORDS: Biquadratic equation with three unknowns, Integral solutions, polygonal and pyramidal numbers, Four Dimensional Figurative numbers, Centered polygonal and pyramidal numbers.

MSC 2000 Mathematics subject classification: 11D25

NOTATIONS:

$T_{m,n}$ -Polygonal number of rank n with size m

P_n^m - Pyramidal number of rank n with size m

SO_n -Stella octangular number of rank n

S_n -Star number of rank n

PR_n - Pronic number of rank n

OH_n - Octahedral number of rank n

J_n -Jacobsthal number of rank of n

j_n - Jacobsthal-Lucas number of rank n

KY_n -keynea number of rank n

$CP_{n,3}$ - Centered Triangular pyramidal number of rank n

$CP_{n,6}$ - Centered hexagonal pyramidal number of rank n

$F_{4,n,5}$ -Four Dimensional Figurative number of rank n whose generating polygon is a pentagon

$F_{4,n,3}$ -Four Dimensional Figurative number of rank n whose generating polygon is a triangle

I. INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular biquadratic Diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity [1-2]. In this context one may refer [3-10] for various problems on the biquadratic Diophantine equations with three variables. This paper concerns with the problem of determining non-trivial integral solutions of the non-homogeneous

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equation with three unknowns given by $x^2-2xy+3y^2=(k^2+2s^2)z^4$. A few relations among the solutions are presented

II. METHOD OF ANALYSIS

The Diophantine equation representing the biquadratic equation with three unknowns is given by

$$x^2-2xy+3y^2=(k^2+2s^2)z^4 \quad (1)$$

Introducing the linear transformations

$$x-y=w \quad (2)$$

in (1) it simplifies to

$$w^2+2y^2=(k^2+2s^2)z^4 \quad (3)$$

The above equation (3) is solved through different approaches and thus, one obtains distinct sets of integer solutions to (1)

A. Case1: k^2+2s^2 is not a perfect square

A.1 Approach1: Let $z=a^2+2b^2$ (4)

Substituting (4) in (3) and using the method of factorisation, define $(w+i\sqrt{2}y)=(k+i\sqrt{2}s)(a+i\sqrt{2}b)^4$

Equating real and imaginary parts, we have

$$w=k(a^4-12a^2b^2+4b^4)-2s(4a^3b-8ab^3)$$

$$y=k(4a^3b-8ab^3)+s(a^4-12a^2b^2+4b^4)$$

In view of (2) and (4), the non-zero distinct integral solutions of (1) are given by

$$\left.\begin{aligned} x&=(k+s)(a^4-12a^2b^2+4b^4)+(k-2s)(4a^3b-8ab^3)\\ y&=k(4a^3b-8ab^3)+s(a^4-12a^2b^2+4b^4)\\ z&=a^2+2b^2 \end{aligned}\right\} \quad (5)$$

A.1.1. Properties:

- ❖ $x(a,a)+y(a,a)+(6s+15k)[24F_{4,a,3}-6CP_{a,6}-11T_{4,a}-12T_{3,a}+6T_{4,a}]=0$
- ❖ $2[3j_{4n}-z(2^{2n},2^{2n})]$ is a nasty number
- ❖ $x(a,a)-y(a,a)+(8s-7k)z(a,a)-(8s-7k)[2T_{3,a^2}+T_{6,a}+2T_{3,a}-T_{4,a}]=0$
- ❖ The following expressions are biquadratic integers:
  (a) $k^3(4x-11y)$
  (b) $k^3(-7x-y)$
- ❖ $x(a,1)+y(a,1)=(k+2s)[2T_{3,a^2}-13T_{4,a}+4]-4(k-s)[SO_a-6T_{3,a}+3T_{4,a}]$

A.2. Approach2: (3) Can be written as

$$w^2+2y^2=(k^2+2s^2)z^4\times 1 \quad (6)$$

(i) Write 1 as

$$1=\frac{((1+i2\sqrt{2})(1-i2\sqrt{2})}{3^2} \quad (7)$$

Using (7) in (6) and employing the method of factorization, define

$$\left(w+i\sqrt{2}y\right)=\frac{(1+i2\sqrt{2})}{3}(k+i\sqrt{2}s)(a+i\sqrt{2}b)^4 \quad (8)$$

Equating real and imaginary parts in (8) we get

$$w=\frac{1}{3}[(k-4s)(a^4-12a^2b^2+4b^4)-2(2k+s)(4a^3b-8ab^3)]$$
$$y=\frac{1}{3}[(k-4s)(4a^3b-8ab^3)+(2k+s)(a^4-12a^2b^2+4b^4)] \qquad (9)$$

Using (9) & (2) and performing some algebra, we get the integral solution of (1) as

$$\left.\begin{aligned} x&=3^3[(3k-3s)(A^4-12A^2B^2+4B^4)-(3k+6s)(4A^3B-8AB^3)]\\ y&=3^3[(k-4s)(4A^3B-8AB^3)+(2k+s)(A^4-12A^2B^2+4B^4)]\\ z&=3^2(A^2+2B^2)\end{aligned}\right\} \qquad (10)$$

(ii) Instead of (7), 1 can also be written as follows:

$$1=\frac{(a^2\mp 2b^2+i2\sqrt{2ab})(a^2\mp 2b^2-i2\sqrt{2ab})}{(a^2\pm 2b^2)^2} \quad \text{(Or)}$$

$$1=\frac{(a\mp 2b+i\sqrt{2}ab)(a\mp 2b-i\sqrt{2}ab)}{(a\pm 2b)^2} \quad \text{(Or)}$$

$$1=\frac{(7+i4\sqrt{2})(7-i4\sqrt{2})}{9^2} \quad \text{(Or)}$$

$$1=\frac{(1+i12\sqrt{2})(1-i12\sqrt{2})}{17^2}$$

Following the same procedure as in approach2, the corresponding integer solutions of (1) can be obtained.

A.3. Approach3:

By assuming $w=w'z,\ y=y'z$ in (3), we get,

$$w'^2+2y'^2=(k^2+2s^2)z^2 \qquad (11)$$

Rewriting (11) as

$$w'^2-k^2z^2=2s^2z^2-2y'^2 \qquad (12)$$

Using the method of factorization, writing (12) as a system of double equations, and using the method of cross multiplication we get two equations in p and q as

$$pw'+2qy'-z(pk+2qs)=0$$
$$qw'-py'+z(kq-ps)=0$$

solving these two equations, we get

$$w'=p^2k+4pqs-2q^2k$$
$$y'=2q^2s-p^2s+2kpq$$
$$z=p^2+2q^2$$

and using (2), the solutions of (1) are obtained as

$$x=[(p^2-2q^2)(k-s)+2pq(2s+k)]z$$
$$y=[(2q^2-p^2)s+2kpq]z$$
$$z=p^2+2q^2$$

B. Case2: k^2+2s^2 is a perfect square

B.1. Approach4:

Choose k and s such that

$$k^2+2s^2=d^2. \qquad (13)$$

By assuming $w=w'zd,\ y=y'zd$ (14)

in (3) and using (13), we get,

whose solution is,

$$w' = \alpha^2 - 2\beta^2, y' = 2\alpha\beta, z = \alpha^2 + 2\beta^2 \qquad (16)$$

By using (15), (14) and (2) we get the integral solutions of (1) as

$$\left. \begin{array}{l} x = d^3[\alpha^4 - 4\beta^4 + 2\alpha\beta(\alpha^2 + 2\beta^2)] \\ y = 2d^3\alpha\beta(\alpha^2 + 2\beta^2) \\ z = d(\alpha^2 + 2\beta^2) \end{array} \right\} \qquad (17)$$

B.2 Approach5*:*

Write (15) as

$$z^2 - w'^2 = 2y'^2, \qquad (18)$$

This can be written as the system of double equations as follows:

Set1: $z + w' = 2y'$, $z - w' = y'$

Set2: $z - w' = 2y'$, $z + w' = y'$

Set3: $z + w' = y'^2$, $z - w' = 2$

Set4: $z - w' = y'^2$, $z + w' = 2$

The system of double equations are solved and using (14) and (2), the corresponding integral solutions to (1) are given by

Set1: $x = 9k^2d, y = 6k^2d, z = 3k$

Set2: $x = 3k^2d, y = 6k^2d, z = 3k$

Set3: $x = d(2k^2 + 1)(2k^2 + 2k - 1), y = 2dk(2k^2 + 1), z = 2k^2 + 1$

Set4: $x = d(1 + 2k^2)(1 - 2k^2 + 2k), y = 2dk(1 + 2k^2), z = 1 + 2k^2$

B.3. Remark:

It is observed that, if (x_0, y_0, z_0) is a given solution to (1), then the triple (x_1, y_1, z_1) also satisfies (1), where $x_1 = n(x_0 - 2y_0), y_1 = -ny_0, z_1 = nz_0$

III. CONCLUSION

The integral solutions of the given ternary biquadratic equation have been discussed and five different patterns were given. One may search for other patterns of solutions and their corresponding properties

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