ISSN: 2350-0328

International Journal of Advanced Research in Science, Engineering and Technology

Vol. 1, Issue 5 , December 2014

Morphisms of Bresets

Nistala V.E.S. Murthy, Lokavarapu Sujatha

Professor, Department of Mathematics, AUCST, Andhra University, Vizag, AP State, India

Research Scholar, Department of Mathematics, AUCST, Andhra University, Vizag, AP State, India

ABSTRACT*:* Several objects like poset,(complete) semi-lattice, (complete) lattice, graph etc., have the underlying object, a set with a binary relation. In this paper morphisms between such sets with binary relations, called bresets, are exclusively studied. Since bresets have an underlying set and a binary relation, notice that for homomorphisms, one can consider both functions and relations on the underlying sets, giving rise to function based homomorphisms and relation based homomorphisms. This paper introduces the notions of function (relational)(co,strong) homomorphism, (weak-co,full) homomorphism and studies lattice algebraic properties of (inverse) images and inverse images of various sub structures of bresets under these homomorphisms in detail.

KEYWORDS*:* Function (relational) (co,strong) homomorphism, (weak-co,full) homomorphism, (inverse) images of sub structures.

A.M.S. subject classification: 05C20,05C63,05C60,97E60.

I. INTRODUCTION

Binary relations are one of the earliest relations known to both mankind and Mathematicians. Basic relations like reflexive, symmetric, transitive, irreflexive, anti-symmetric, cyclic, etc. are all binary relations which play an important role in the studies of several order-structures like semi ordered set, well ordered set, totally ordered set, partial ordered set and higher-order structures like, join (meet) semi lattice, (distributive, modular, deMorgan) lattice, (join,meet) and still higher order structures like complete lattice, infinite (join,meet) distributive lattice, completely distributive complete lattice etc.. Notice that the under lying object for all of them is a set with a binary relation.

Another important class of objects, on a set with a binary relation, is the graph which is nothing but a finite set (of nodes) together with a binary relation (of edges). The Theory of Graphs is well known for its applications both in Hardware and Software of Computer Science.

However, a set with a binary operation, as an object by itself is not exclusively studied and primarily this aspect was taken up in Nistala and Lokavarapu[15]. We call a set with a binary relation a breset (Binarily RElated SET) and in this paper we study various types of morphisms between these objects. Since a digraph is a finite set with a (finite) binary relation and since the notion of a breset has no restriction of finiteness either on the number of elements of the underlying set or on the size of the binary relation, a breset can be regarded as an infinite digraph and hence a generalization of digraph. Of course, infinite (di) graphs were studied in conjunction with groups and/or vector spaces. To see some work in this direction, one can refer to Finucane[7], Seifter[18], Soardi and Woess[19], Bondy and Hemminger[3], Andreae[2] etc..

In Nistala and Lokavarapu[16], we extended such notions of (di) graphs as conjunctive product (also known as tensor or categorical product), disjunctive product (also known as co-normal product) to *arbitrary* families of bresets and studied them. Also we introduced and studied such notions as factors, radicals for bresets and proved such results as: for a family of bresets $(\mathsf{A}_i)_{i\in I}$, (a) A_i is reflexive (symmetric, transitive, anti-symmetric) for all $i \in I$ if and only if $\prod^{c}_{i\in I}\mathsf{A}_i$ is reflexive (symmetric, transitive, anti-symmetric) when ever each A_i is non empty for all $i \in I$ under some simple different conditions and (lattice) algebraic properties of (inverse) images of sub structures of bresets under (function and relation) morphisms between bresets etc. all of which are more of purely mathematical interest. *In fact, it is for these reasons that we preferred the word breset over the word infinite digraph.*

ISSN: 2350-0328

International Journal of Advanced Research in Science, Engineering and Technology

Vol. 1, Issue 5 , December 2014

Since bresets have an underlying set and a binary operation, notice that for homomorphisms, one can consider both functions and relations on the underlying sets, giving rise to function based homomorphisms and relation based homomorphisms.

In this paper, the notions of function (relational)(co,strong) homomorphism, (weak-co,full) homomorphism are introduced and properties of images and inverse images of various sub structures under these homomorphisms are studied in detail.

Notice that the notions of function homomorphism and relation homomorphism between binary and in general n-ary relations are well known. However, other notions which we study for them in this paper are new. In the second section we recall some preliminaries which are used in subsequent sections; in the third section we introduce the notions of function homomorphism, function co-homomorphism, weak function co-homomorphism, strong function homomorphism, full function homomorphism and study the properties (inverse) images of various sub structures of bresets under these homomorphisms and in the fourth section, we introduce the notions of relational homomorphism, relational co-homomorphism and strong relational homomorphism and study the properties (inverse) images of various sub structures of bresets under these homomorphisms.

Note that homomorphisms in graph theory are an important part of study because (a) a graph G is k-colorable iff there is a homomorphism from G to C_k, the complete graph on k-vertices (b) in general where H is a graph, an H-colouring of a graph G amounts to finding a homomorphism of G in to H.

II. PRELIMINARIES

In this section the notions of breset, lower sub breset, upper sub breset and sub breset are recalled and some relations between them are stated along with some examples and counter examples.

A breset is any ordered pair (A, R), where A is called the *underlying set* or shortly the *u-set* of (A, R) and R is a binary relation on A. Let us recall that a binary relation R on a set A is any subset of $A \times A$ (b) for any pair of bresets (A, R), (B, S), (A, R) is *equal* to (B, S), denoted by (A, R) = (B, S), if and only if A = B and R = S (c) A breset (A, R) is *empty breset* or simply *empty*, denoted by Φ, iff the underlying set A = ϕ and the binary relation R = ϕ (d) Clearly, a breset (A, R) is empty if and only if A = ϕ. Further, (i) in a breset (A, R), it can so happen that the u-set A is nonempty but the binary relation R on A is empty and (ii) there can be several bresets with the same u-set A.

Notation: Since a breset (A, R) is uniquely determined by its binary relation R on the set A, here onwards for notation convenience, we denote the breset (A, R) by A and the binary relation R by β_{A}. Also, Whenever *further, through out this and other chapters on bresets, the script letters and the suffixed script letters always stand for the bresets. In other words, the* P *stands for the ordered pair* (P, β_{P}), *etc..*

Let A, B be a pair of bresets.
(a) A is said to be a *sub system* of B iff $A \subseteq B$ (b) A is said to be a *lower sub breset* or simply l-sub breset of B iff $A \subseteq B$ and $\beta_{\mathsf{A}} \subseteq \beta_{\mathsf{B}} \cap (A \times A)$ (c) for any breset X, the set of all l-sub bresets of X is denoted by $\mathsf{S}_l(\mathsf{X})$ (d) A is said to be a *upper sub breset* or simply u-sub breset of B iff $A \subseteq B$ and $\beta_{\mathsf{A}} \supseteq \beta_{\mathsf{B}} \cap (A \times A)$ (e) for any breset X, the set of all upper sub bresets of X is denoted by $\mathsf{S}_u(\mathsf{X})$ (f) A is said to be a *sub breset* of B iff $A \subseteq B$ and β_{A} = $\beta_{\mathsf{B}} \cap (A \times A)$ (g) for any breset X, the set of all sub bresets of X is denoted by $\mathsf{S}(\mathsf{X})$.
Clearly, when the underlying set A of the breset A is finite, the notion of l-sub breset of a breset is equivalent to the notion of sub digraph of a digraph and the notion of sub breset of a breset is equivalent to the notion of induced sub di-graph.

ISSN: 2350-0328

International Journal of Advanced Research in Science, Engineering and Technology

Vol. 1, Issue 5 , December 2014

For any breset X, (1) being l-sub breset is a binary relation on $\mathsf{S}_l(\mathsf{X})$ making it a breset and further, a poset (2) being u-sub breset is a binary relation on $\mathsf{S}_u(\mathsf{X})$ making it a breset and further, a poset (3) being sub breset is a binary relation on $\mathsf{S}(\mathsf{X})$ making it a breset and further, a poset.
Clearly, (1) every sub breset is a lower (upper) sub breset (2) an l-sub breset need *not* be a sub breset (3) a u-sub breset need *not* a sub breset (4) for any breset B and for any subset A of B, the subset $\beta_{\mathsf{A}} = \beta_{\mathsf{B}} \cap (A \times A)$ of $B \times B$, is such that A is always a sub breset of B, called the induced sub breset. The counter examples are easy construct and hence left.

III. FUNCTION BASED HOMOMORPHISM

As mentioned earlier, in this section we introduce the notions of function homomorphism, function co-homomorphism, weak function co-homomorphism, strong function homomorphism, full function homomorphism and study the properties (inverse) images of various sub structures of bresets under these homomorphisms.

Definition 3.1 *For any pair of bresets* X *and* Y *and for any map* $f: X \to Y$, f *is a function homomorphism or simply homomorphism of bresets* X *in to* Y, *denoted again by* $f: \mathsf{X} \to \mathsf{Y}$, *iff* $(a,b) \in \beta_{\mathsf{X}}$ *implies* $(fa, fb) \in \beta_{\mathsf{Y}}$.

Remark: When the underlying sets X and Y of the bresets X and Y are finite, the function homomorphism of bresets above is simply the homomorphism of di-graphs (cf..P676, Jorgen-Gregory[10]).
Definition 3.2 *For any pair of bresets* X *and* Y *and for any map* $f: X \to Y$, f *is a function co-homomorphism or simply co-homomorphism of bresets* X *in to* Y, *again denoted by* $f: \mathsf{X} \to \mathsf{Y}$, *iff* $(fx, fy) \in \beta_{\mathsf{Y}}$ *implies* $(x, y) \in \beta_{\mathsf{X}}$.

Definition 3.3 *For any pair of bresets* X *and* Y *and for any map* $f: X \to Y$, f *is a weak function co-homomorphism or simply weak co-homomorphism of bresets* X *in to* Y, *again denoted by* $f: \mathsf{X} \to \mathsf{Y}$, *iff* $(fx, fy) \in \beta_{\mathsf{Y}}$ *implies there exists* $(a,b) \in \beta_{\mathsf{X}}$ *such that* $fa = fx$ *and* $fb = fy$.

Definition 3.4 *For any pair of bresets* X *and* Y *and for any map* $f: X \to Y$, f *is a strong function homomorphism or simply strong homomorphism of bresets* X *in to* Y, *again denoted by* $f: \mathsf{X} \to \mathsf{Y}$, *iff* $(fx, fy) \in \beta_{\mathsf{Y}}$ *iff* $(x, y) \in \beta_{\mathsf{X}}$. *In other words,* f *is both homomorphism and a co-homomorphism.*

Definition 3.5 *For any pair of bresets* X *and* Y *and for any map* $f: X \to Y$, f *is a full function homomorphism or simply full-homomorphism of bresets* X *in to* Y, *again denoted by* $f: \mathsf{X} \to \mathsf{Y}$, *iff* f *is homomorphism and weak co-homomorphism.*
Clearly, co-homomorphism implies weak co homomorphism and strong homomorphism implies full homomorphism. The converses, however, are *not* true.
In what follows, we give examples and counter examples indicating (in)depen-dence of the above notions.
Example 3.6 *Let* $\mathsf{X} = \{a,b,c,d\}$, $\beta_{\mathsf{X}} = \{ab\}$, $\mathsf{Y} = \{p,q,r,s\}$, $\beta_{\mathsf{Y}} = \{pq, rs\}$, $f: X \to Y$, $f = \{ap, bq, cr, ds\}$. *Then* $ab \in \beta_{\mathsf{X}}$ *implies* $fafb = pq \in \beta_{\mathsf{Y}}$.
Then f is a homomorphism, but *not* weak co-homomorphism because $rs = fcfd \in \beta_{\mathsf{Y}}$ but there does *not* exists $uv \in \beta_{\mathsf{X}}$ such that $fu = r$, $fv = s$.

Example 3.7 *Let* $\mathsf{X} = \{a,b,c,d\}$, $\beta_{\mathsf{X}} = \{cd\}$, $Y = \{p,q\}$, $\beta_{\mathsf{Y}} = \{pq\}$, $f: X \to Y$, $f = \{ap, bq, cp, dq\}$.

ISSN: 2350-0328

International Journal of Advanced Research in Science, Engineering and Technology

Vol. 1, Issue 5 , December 2014

Then f is a homomorphism and weak co-homomorphism and hence a full homomorphism. But f is *not* a strong homomorphism because f is not co-homomorphism or $fafb = pq \in \beta_Y$ but $ab \notin \beta_X = cd$.

Example 3.8 *Let* $X = \{a,b,x,y\}$, $\beta_X = \{ab, ax\}$, $Y = \{p,q\}$, $\beta_Y = \{pq\}$, $f: X \to Y$, $f = \{xp, yq, ap, bq\}$.

Then f is a weak co-homomorphism, but *not* a homomorphism because $ax \in \beta_X$ but $fafx \notin \beta_Y$ and *not* a co-homomorphism because $fxfy \in \beta_Y$ but $xy \notin \beta_X$.

Example 3.9 *Let* $X = \{a,b,x,y\}$, $\beta_X = \{ab, ax\}$, $Y = \{p,q\}$, $\beta_Y = \{pq, pp\}$, $f: X \to Y$, $f = \{xp, yq, ap, bq\}$.

Then f is a weak co-homomorphism and a homomorphism because $ax \in \beta_X$, $fafx \in \beta_Y$. So, f is a full homomorphism but *not* a strong homomorphism because f is *not* a co-homomorphism because $fxfy \in \beta_Y$ but $xy \notin \beta_X$.

Example 3.10 *Let* $X = \{a,b\}$, $\beta_X = \{aa, ab\}$, $Y = \{p,q\}$, $\beta_Y = \{pq\}$, $f: X \to Y$, $f = \{ap, bq\}$. Then f is co-homomorphism but *not* homomorphism because $aa \in \beta_X$ but $fafa = pp \notin \beta_Y = pq$.

Lemma 3.11 *For any pair of bresets* X, Y *and for any map* $f: X \to Y$, $f: X \to Y$ *is a co-homomorphism iff for all* $x, y \in X$, $(fx, fy) \in \beta_Y$ *implies* $f^{-1}fx \times f^{-1}fy \subseteq \beta_X$.

Proof: ($\Rightarrow$:) Since $f: X \to Y$ is co-homomorphism $(fa, fb) \in \beta_Y$ implies $(a,b) \in \beta_X$ for all $a, b \in X$. Let $x, y \in X$ be such that $(fx, fy) \in \beta_Y$. we show that $f^{-1}fx \times f^{-1}fy \subseteq \beta_X$ Let $(a,b) \in f^{-1}fx \times f^{-1}fy$. Then $(fa, fb) = (fx, fy) \in \beta_Y$. Since f is a co-homomorphism, $(a,b) \in \beta_X$, so that $f^{-1}fx \times f^{-1}fy \subseteq \beta_X$.

($\Leftarrow$:) Let $(fx, fy) \in \beta_Y$ implies $(x,y) \in f^{-1}fx \times f^{-1}fy \subseteq \beta_X$ implies $(x,y) \in \beta_X$.

Definition 3.12 *For any pair of bresets* X, Y, *for any map* $f: X \to Y$ *and for any sub system* U *of* X, *the breset* V *where* $V = fU$ *and* $\beta_V = \{(fp, fq) \mid (p,q) \in \beta_U\}$ *is called the image sub system of* U *and is denoted by* fU. *In other words,* $\beta_{fU} = \{(fp, fq) \mid (p,q) \in \beta_U\}$.

Note: whenever U is a sub system of X, fU is a sub system of Y.

Observe that V is a well defined breset because if $(p,q) \in \beta_V \subseteq U^2$ then $(fp, fq) \in (fU)^2 = V^2$, so that $\beta_V \subseteq V^2$.

Lemma 3.13 *Let* X, Y *be bresets,* $f: X \to Y$ *homomorphism,* U *is an l-sub breset of* X *implies* fU *is an l-sub breset of* Y.

Proof: Since U is an l-sub breset of X, $U \subseteq X$ and $\beta_U \subseteq \beta_X \cap U^2$. by the definition of l-sub breset, it is enough to show that $\beta_{fU} \subseteq \beta_Y$. Let $\alpha \in \beta_{fU}$. Then $\alpha = (fp, fq)$, $(p,q) \in \beta_U \subseteq \beta_X$. Since f is a homomorphism and $(p,q) \in \beta_X$, $(fp, fq) \in \beta_Y$, as required.

ISSN: 2350-0328

International Journal of Advanced Research in Science, Engineering and Technology

Vol. 1, Issue 5 , December 2014

Definition 3.14 *For any pair of bresets* X, Y, *for any homomorphism* $f : X \to Y$ *and for any l-sub breset* U *of* X, *the l-sub breset* fU *defined as in the previous lemma is called the image l-sub breset of* U *under* f.

Lemma 3.15 *Let* X, Y *be bresets and* $f : X \to Y$ *be a strong homomorphism. Then* U *is a sub breset of* X *implies* fU *is a sub breset of* Y.

Proof: Since U is a sub breset of X, U is an l-sub breset of X and since f is a strong homomorphism, it is a homomorphism. So, by Lemma 3.1.13 fU is a l-sub breset of Y. Hence it is enough to show that $\beta_Y \cap (fU)^2 \subseteq \beta_{fU}$. Let $\alpha \in \beta_Y \cap (fU)^2$. Then $\alpha = (fp, fq)$, $(p,q) \in U^2$. Since f is a strong homomorphism and $\alpha = (fp, fq) \in \beta_Y$, we get that $(p,q) \in \beta_X$. Since U is a sub breset of X, $\beta_U = \beta_X \cap U^2$, from the above we get that $(p,q) \in \beta_X \cap U^2 = \beta_U$ and from the definition of β_{fU}, $\alpha = (fp, fq) \in \beta_{fU}$, so that $\beta_Y \cap (fU)^2 \subseteq \beta_{fU}$.

Definition 3.16 *For any pair of bresets* X, Y, *for any strong homomorphism* $f : X \to Y$ *and for any sub breset* U *of* X, *the sub breset* fU *defined as in the previous lemma is called the image sub breset of* U *under* f.

Lemma 3.17 *Let* X, Y *be bresets and* $f : X \to Y$ *be a co-homomorphism. Then* U *is a u-sub breset of* X *implies* fU *is a u-sub breset of* Y.

Proof: By the definition of u-sub breset, it is enough to show that $\beta_{fU} \supseteq \beta_Y \cap (fU)^2$. Let $\alpha \in \beta_Y \cap (fU)^2$. Then $\alpha = (fp, fq)$, $(p,q) \in U^2$. Since f is a co-homomorphism and $\alpha = (fp, fq) \in \beta_Y$, we get that $(p,q) \in \beta_X$. Since U is a u-sub breset of X, $\beta_U \supseteq \beta_X \cap U^2$, we get that $(p,q) \in \beta_X \cap U^2 \subseteq \beta_U$ and by the definition of β_{fU}, $\alpha = (fp, fq) \in \beta_{fU}$, so that $\beta_Y \cap fU^2 \subseteq \beta_{fU}$.

Definition 3.18 *For any pair of bresets* X, Y, *for any co-homomorphism* $f : X \to Y$ *and for any u-sub breset* U *of* X, *the u-sub breset* fU *defined as in the previous lemma is called the image u-sub breset of* U *under* f.

The following four results actually explain the relations between various function homomorphisms and the inclusion maps.

Lemma 3.19 *Let* X, Y *be bresets such that* X *is a sub system of* Y. *Then* X *is an l-sub breset of* Y *if and only if the inclusion map* $i : X \to Y$ *is the homomorphism* $i : X \to Y$.

Proof: ($\Rightarrow$:) Since X is an l-sub breset, we have $\beta_X \subseteq \beta_Y \cap X^2$. Let $(a,b) \in \beta_X$. Since $\beta_X \subseteq \beta_Y, (ia, ib) = (a,b) \in \beta_Y$, so that i is a homomorphism.

($\Leftarrow$:) Let $i : X \to Y$ be homomorphism. Let $(a,b) \in \beta_X$. Since i is a homomorphism, $(a,b) = (ia, ib) \in \beta_Y$. So that $\beta_X \subseteq \beta_Y$ and so $\beta_X \subseteq \beta_Y \cap X^2$.

Lemma 3.20 *Let* X, Y *be bresets such that* X *is a sub system of* Y. *Then* X *is a u-sub breset of* Y *if and only if the inclusion map* $i : X \to Y$ *is the co-homomorphism* $i : X \to Y$.

Proof: ($\Rightarrow$:) Let $a, b \in X$ be such that $(ia, ib) \in \beta_Y$. Then $(a,b) = (ia, ib) \in \beta_Y$. Since X is a u-sub breset of Y, $\beta_X \supseteq \beta_Y \cap X^2$. Since $(a,b) \in X^2$, from the above we get that $(a,b) \in \beta_Y \cap X^2 \subseteq \beta_X$, as required, so that i is a co-homomorphism.

($\Leftarrow$:) Let $i: \mathsf{X} \to \mathsf{Y}$ be a co-homomorphism. By the definition of u-sub breset, it is enough to show that $\beta_{\mathsf{X}} \supseteq \beta_{\mathsf{Y}} \cap X^2$. Let $\alpha \in \beta_{\mathsf{Y}} \cap X^2$. Then $\alpha \in \beta_{\mathsf{Y}}$ and $\alpha = (a,b)$, $(a,b) \in X^2$. Since i is a co-homomorphism and $(a,b) = (ia,ib) \in \beta_{\mathsf{Y}}$, $(a,b) \in \beta_{\mathsf{X}}$, as required.

Lemma 3.21 *Let X, Y be bresets such that X is a sub system of Y. Then X is a u-sub breset of Y if and only if the inclusion map $i: X \to Y$ is the weak co-homomorphism $i: \mathsf{X} \to \mathsf{Y}$.*

Proof: ($\Rightarrow$:) Since co-homomorphism is a weak co-homomorphism, we are done by Lemma 3.12.

($\Leftarrow$:) Let $i: \mathsf{X} \to \mathsf{Y}$ be a weak co-homomorphism. By the definition of u-sub breset, it is enough to show that $\beta_{\mathsf{X}} \supseteq \beta_{\mathsf{Y}} \cap X^2$. Let $\alpha \in \beta_{\mathsf{Y}} \cap X^2$. Then $\alpha \in \beta_{\mathsf{Y}}$ and $\alpha = (a,b)$, $(a,b) \in X^2$. Since i is a weak co-homomorphism and $(ia,ib) = (a,b) \in \beta_{\mathsf{Y}}$, we get $p,q \in X$ such that $ip = ia$, $iq = ib$. But then $\alpha = (a,b) = (p,q) \in \beta_{\mathsf{X}}$, as required.

Corollary 3.22 *Let X, Y be bresets such that X is a sub system of Y. Then X is a sub breset of Y if and only if the inclusion map $i: X \to Y$ is the strong homomorphism $i: \mathsf{X} \to \mathsf{Y}$ if and only if the inclusion map $i: X \to Y$ is the full homomorphism $i: \mathsf{X} \to \mathsf{Y}$.*

Definition 3.23 *For any pair of bresets X, Y and for any sub system V of Y, the breset U of X such that $U = f^{-1}V$, $\beta_{\mathsf{U}} = \{(a,b) \in \beta_{\mathsf{X}} \mid (fa, fb) \in \beta_{\mathsf{V}}\}$ is called the inverse image of the sub system V of Y and is denoted by $f^{-1}\mathsf{V}$.*

In other words, $\beta_{f^{-1}\mathsf{V}} = \{(a,b) \in \beta_{\mathsf{X}} \mid (fa, fb) \in \beta_{\mathsf{V}}\}$.

Lemma 3.24 *For any pair of bresets X, Y and for any map $f: \mathsf{X} \to \mathsf{Y}$ is a co-homomorphism, V is an l-sub breset of Y implies $f^{-1}\mathsf{V}$ is an l-sub breset of X.*

Proof: Since V is an l-sub breset of Y, $V \subseteq Y$ and $\beta_{\mathsf{V}} \subseteq \beta_{\mathsf{Y}} \cap (V \times V)$. By the definition of l-sub breset it is enough to show that $\beta_{f^{-1}\mathsf{V}} \subseteq \beta_{\mathsf{X}}$. Let $\alpha \in \beta_{f^{-1}\mathsf{V}}$. Then $\alpha = (a,b)$ and $(fa, fb) \in \beta_{\mathsf{V}}$. Since f is a co-homomorphism and $(fa, fb) \in \beta_{\mathsf{V}} \subseteq \beta_{\mathsf{Y}}$, we get that $\alpha = (a,b) \in \beta_{\mathsf{X}}$, as required.

Lemma 3.25 *For any pair of bresets X, Y and for any map $f: \mathsf{X} \to \mathsf{Y}$ is a homomorphism, V is a u-sub breset of Y implies $f^{-1}\mathsf{V}$ is a u-sub breset of X.*

Proof: We need to show that $\beta_{f^{-1}\mathsf{V}} \supseteq \beta_{\mathsf{X}} \cap (f^{-1}V)^2$. Let $\alpha \in \beta_{\mathsf{X}} \cap (f^{-1}V)^2$. Then $\alpha \in \beta_{\mathsf{X}}$ and $\alpha \in \beta_{\mathsf{X}}$ and $\alpha = (a,b) \in (f^{-1}V)^2$. Since f is a homomorphism and $(a,b) = \alpha \in \beta_{\mathsf{X}}$, we have $(fa, fb) \in \beta_{\mathsf{Y}}$. Since $(a,b) \in (f^{-1}V)^2$, $(fa, fb) \in V^2$, so that $(fa, fb) \in \beta_{\mathsf{Y}} \cap V^2 \subseteq \beta_{\mathsf{V}}$, since V is a u-sub breset of Y. Clearly by the definition of $f^{-1}\mathsf{V}$, $\alpha = (a,b) \in \beta_{f^{-1}\mathsf{V}}$, as required.

Lemma 3.26 *For any pair of bresets X, Y and for any mapping $f: \mathsf{X} \to \mathsf{Y}$ is a strong homomorphism, V is a sub breset of Y implies $f^{-1}\mathsf{V}$ is a sub breset of X.*

Proof: (i) Since V is a sub breset, we have V is a l-sub breset and since f is a strong homomorphism we have f is a co-homomorphism, it follows by Lemma 3.16 that $f^{-1}\mathsf{V}$ is a l-sub breset of X.

ISSN: 2350-0328
International Journal of Advanced Research in Science, Engineering and Technology
Vol. 1, Issue 5 , December 2014

(ii) Since V is a sub breset, we have V is a u-sub breset and since f is a strong homomorphism we have f is a homomorphism, it follows by Lemma 3.17 that $f^{-1}\mathsf{V}$ is a u-sub breset of X.
Since a sub system of a breset is a sub breset if and only if it is both a l-sub breset and u-sub breset, from (i) and (ii) above we get that $f^{-1}\mathsf{V}$ is a sub breset of X.

IV. RELATION BASED HOMOMORPHISM

In this section, we introduce the notions of relational homomorphism, relational co-homomorphism and strong relational homomorphism and study the properties (inverse) images of various sub structures of bresets under these homomorphisms.

Definition 4.1 *For any pair of bresets* X,Y *and for any binary relation* $R: X \to Y$, R *is a relational homomorphism of bresets from* X *into* Y, *denoted again by* $R:\mathsf{X} \to \mathsf{Y}$, *iff* $(a,b) \in \beta_{\mathsf{X}}$ *implies* $Ra \times Rb \subseteq \beta_{\mathsf{Y}}$, *where* $Rc = \{d \in Y \mid cd \in R\}$, *the image of* $c \in X$ *under* R.

Definition 4.2 *For any pair of bresets* X,Y *and for any relation* $R: X \to Y$, R *is a relational co-homomorphism of bresets* X *into* Y, *denoted again by* $R:\mathsf{X} \to \mathsf{Y}$, *iff* $(c,d) \in \beta_{\mathsf{Y}}$ *implies* $R^{-1}c \times R^{-1}d \subseteq \beta_{\mathsf{X}}$.

Notice that R is a relational co-homomorphism iff the inverse relation $R^{-1}: Y \to X$ given by R^{-1} = $\{(y,x) \in Y \times X \mid (x,y) \in R\}$ is a ralational homomorphism.

Definition 4.3 *For any pair of bresets* X,Y *and for any relation* $R: X \to Y$, R *is a strong relational homomorphism of bresets* X *into* Y, *denoted again by* $R:\mathsf{X} \to \mathsf{Y}$, *iff* R *is both a relational homomorphism and a relational co-homomorphism.*

Clearly, a strong relational homomorphism is both a relational homomorphism and a relational co-homomorphism.

In what follows, we give examples to show that (a) a relational homomorphism is *not* necessarily a relational co-homomorphism and hence *not* a strong relational homomorphism (b) a relational co-homomorphism is *not* necessarily a relational homomorphism and hence *not* a strong relational homomorphism either.

Example 4.4 *Let* $X = \{a,b\}$, β_{X} = $\{aa,ab\}$, $Y = \{p,q,r\}$, β_{Y} = $\{pq,pr,qq,qr,pp,qp,rq,rr\}$, $f: X \to Y$, R = $\{ap,,aq,bq,br\}$, Ra = $\{p,q\}$, Rb = $\{q,r\}$, $Ra \times Ra$ = $\{pp,pq,qp,qq\}$ *and* $Ra \times Rb$ = $\{pq,pr,qq,qr\}$.

Clearly, R is a relational homomorphism. However, $Rb \times Rb \subseteq \beta_{\mathsf{Y}}$ but $bb \notin \beta_{\mathsf{X}}$ so that R is *not* a relational co-homomorphism and hence *not* a strong relational homomorphism.

Example 4.5 *Let* X = $\{a,b\}$, β_{X} = $\{aa,ab\}$, Y = $\{p,q,r\}$, β_{Y} = $\{pq,qr\}$, $R: X \to Y$, R = $\{ap,,aq,bq,br\}$, Ra = $\{p,q\}$, Rb = $\{q,r\}$, $Ra \times Ra \subset \beta_{\mathsf{Y}}$, $Ra \times Rb \subset \beta_{\mathsf{Y}}$, $Rb \times Ra \subset \beta_{\mathsf{Y}}$ *and* $Rb \times Rb \subset \beta_{\mathsf{Y}}$.

Clearly, R is trivially a relational co-homomorphism but *not* a relational homomorphism and hence *not* a strong relational homomorphism.

Example 4.6 *Let* X = $\{a,b\}$, β_{X} = $\{aa,ab,bb\}$, Y = $\{p,q,r\}$, β_{Y} = $\{pq,pr,qq,qr,pp,qp,rq,rr\}$, $R: X \to Y$, R = $\{ap,,aq,bq,br\}$, Ra = $\{p,q\}$, Rb = $\{q,r\}$, $Ra \times Ra$ = $\{pp,pq,qp,qq\}$ *and* $Ra \times Rb$ = $\{pq,pr,qq,qr\}$.

ISSN: 2350-0328
International Journal of Advanced Research in Science, Engineering and Technology
Vol. 1, Issue 5 , December 2014

Clearly, R is a relational homomorphism. However, $Rb \times Rb \subseteq \beta_{Y}$ and $bb \in \beta_{X}$ so that R is a relational co-homomorphism. Therefore R is a strong relational homomorphism.

Lemma 4.7 *Let* X, Y *be bresets,* $R: X \to Y$ *be a relational homomorphism and* U *be a sub system of* X. *Then* U *is an l-sub breset of* X *implies* RU *is an l-sub breset of* Y.

Proof: By the definition of l-sub breset, we need to show that $\beta_{RU} \subseteq \beta_{Y} \cap (RU)^2$. Since $\beta_{RU} \subseteq (RU)^2$, it is enough to show that $\beta_{RU} \subseteq \beta_{Y}$. So, let $\alpha \in \beta_{RU}$ implies $\alpha = (y_1, y_2) \in Y^2$ such that there exists $(x_1, x_2) \in \beta_{U}$, $(x_i, y_i) \in R$, $i = 1,2$. Since U is a l-sub breset of X, $Rx_1 \times Rx_2 \subseteq \beta_{Y}$. Since $(x_1, y_1) \in R$ and $(x_2, y_2) \in R$, we have $(y_1, y_2) \in Rx_1 \times Rx_2 \subseteq \beta_{Y}$, so that $\alpha = (y_1, y_2) \in \beta_{Y}$ as required.

Lemma 4.8 *Let* X, Y *be bresets,* $R: X \to Y$ *be a relational co-homomorphism and* U *be a sub system of* X. *Then* U *is a u-sub breset of* X *implies* RU *is a u-sub breset of* Y.

Proof: By the definition of u-sub breset, we need to show that $\beta_{RU} \supseteq \beta_{Y} \cap (RU)^2$. Let $\alpha \in \beta_{Y} \cap (RU)^2$. Then $\alpha \in \beta_{Y}$ and since $\alpha \in (RU)^2$, we have $\alpha = (b_1, b_2) \in Y^2$ such that there exists $a_1, a_2 \in U$, $(a_1, b_1), (a_2, b_2) \in R$. To show $(b_1, b_2) \in \beta_{RU}$, it is enough to show that $(a_1, a_2) \in \beta_{X}$. Since R is a Relational co-homomorphism, $(y_1, y_2) \in \beta_{Y}$ implies $R^{-1}b_1 \times R^{-1}b_2 \subseteq \beta_{X}$. Since $(a_1, b_1), (a_2, b_2) \in R$, $(a_1, a_2) \in R^{-1}b_1 \times R^{-1}b_2 \subseteq \beta_{X}$. Now since U is a u-sub breset of X, $\beta_{U} \supseteq \beta_{X} \cap U^2$. So, $(a_1, a_2) \in \beta_{U}$ as required.

Lemma 4.9 *Let* X, Y *be bresets,* $R: X \to Y$ *be a relational strong homomorphism and* U *be a sub system of* X. U *is a sub breset of* X *implies* RU *is a sub breset of* Y.

Proof: Let us recall that for any pair of bresets X, Y and for any sub system U of X, then the following are true:

(1) If $R: X \to Y$ is a relational homomorphism and U is a l-sub breset of X. Then RU is a l-sub breset of Y.

(2) $R: X \to Y$ is a relational co-homomorphism and U is a u-sub breset of X. Then RU is a u-sub breset of Y.

(3) U is a sub breset of X if and only if U is a l-sub breset of X and U is a u-sub breset of X.

(4) R is a relational strong homomorphism if and only if it is a relational homomorphism and a relational co-homomorphism.

From the above, it follows that RU is a sub breset of Y.

Lemma 4.10 *Let* X, Y *be bresets,* V *be a sub system of* Y *and* $R: X \to Y$ *be a relational co-homomorphism. Then if* V *is an l-sub breset of* Y, *then* $R^{-1}V$ *is an l-sub breset of* X.

Proof: Since V is a l-sub breset of Y, $\beta_{V} \subseteq \beta_{Y} \cap V^2$. By the definition of l-sub breset, we need to show $\beta_{R^{-1}V} \subseteq \beta_{X} \cap (R^{-1}V)^2$. Since $\beta_{R^{-1}V} \subseteq (R^{-1}V)^2$, it is enough to show that $\beta_{R^{-1}V} \subseteq \beta_{X}$. Let $\alpha \in \beta_{R^{-1}V}$. Then $\alpha = (x_1, x_2) \in X^2$ where there exists $(y_1, y_2) \in \beta_{V} \subseteq \beta_{Y}$, we have $R^{-1}y_1 \times R^{-1}y_2 \subseteq \beta_{X}$. Since $(x_1, y_1), (x_2, y_2) \in R$, $(x_1, x_2) \in R^{-1}y_1 \times R^{-1}y_2 \subseteq \beta_{X}$, so that $\alpha = (x_1, x_2) \in \beta_{X}$ as required.

Lemma 4.11 *Let* X, Y *be bresets,* V *be a sub system of* Y *and* $R: X \to Y$ *be a relational co-homomorphism. Then if* V *is a l-sub breset of* Y, *then* $R^{-1}V$ *is a l-sub breset of* X.

Proof: By the definition of u-sub breset, we need to show $\beta_{R^{-1}V} \supseteq \beta_{X} \cap (R^{-1}V)^2$. Let $\alpha \in \beta_{X} \cap (R^{-1}V)^2$. Then $\alpha \in \beta_{X}$ and $\alpha \in (R^{-1}V)^2$ from which we get that $\alpha = (x_1, x_2) \in X^2$, there exists $(y_1, y_2) \in V^2$, such that

$(x_1, y_1), (x_2, y_2) \in R$. Since R is a relational homomorphism and $(x_1, x_2) \in \beta_{\mathsf{X}}$, we have $Rx_1 \times Rx_2 \subseteq \beta_{\mathsf{Y}}$. But from $(x_1, y_1), (x_2, y_2) \in R$, we have $(y_1, y_2) \in Rx_1 \times Rx_2 \subseteq \beta_{\mathsf{Y}}$ so that $(y_1, y_2) \in \beta_{\mathsf{Y}} \cap V^2 \subseteq \beta_{\mathsf{V}}$, the last containment is due to V being u-sub breset of Y.

Lemma 4.12 *Let X, Y be bresets, U be a sub system of X and $R : \mathsf{X} \to \mathsf{Y}$ be a relational strong homomorphism. Then if V is a sub breset of Y, then $R^{-1}\mathsf{V}$ is a sub breset of X.*

Proof: Let us recall that for any pair of bresets X, Y and for any sub system U of X, then the following are true:

(1) If $R : \mathsf{X} \to \mathsf{Y}$ is a relational co-homomorphism and V is a l-sub breset of Y. Then $R^{-1}\mathsf{V}$ is a l-sub breset of X.

(2) $R : \mathsf{X} \to \mathsf{Y}$ is a relational homomorphism and V is a u-sub breset of Y. Then $R^{-1}\mathsf{V}$ is a u-sub breset of X.

(3) V is a sub breset of Y if and only if V is a l-sub breset of Y and V is a u-sub breset of Y.

(4) R is a relational strong homomorphism if and only if it is a relational homomorphism and a relational co-homomorphism.

From the above, it follows that $R^{-1}\mathsf{V}$ is a sub breset of X.

V. CONCLUSION

More order properties of (inverse) image induced maps between the three collections of substructures of bresets and a similar study for relational algebras is under study. Further, a categorical study of the categories of realtions (relational algebras) together with function based and relation based morphisms can be made.

REFERENCES

[1] Adamek, J., Herrlich, H., and Strecker, G.E., Abstract And Concrete Categories The Joy Of Cats.
[2] Andreae, T., On the reconstruction of locally finite, infinite graphs, J. combin. Inform. System Sci. 7 (1982) 65-74.
[3] Bondy, J.A. and Hemminger, R.L.,Reconstructing infinite graphs, pacific J. Math. 52 (1974) 331-340.
[4] Birkhoff, G., Lattice Theory, Amer. Math. Soc., Providence, R. I., Colloquium publications.
[5] Charles Desoer and Ennest Kuh, Basic Circuit Theory, McGraw Hill, 1969.
[6] Diestel, R.,Graph Theory, Electronic Edition 2000, Springer-Verlag New York 1997, 2000.
[7] Finucane, H., Finite voronoi dcompositions of infinite vertex transitive graphs. (arxiv.org/pdf/ 1111.0472v1.pdf).
[8] Herrlich, H., and Strecker, G.E., Categori Theory : an introduction, Allyn and Bacon 1973.
[9] Joe L. Mott, Abraham Kandel and Theodore P. Baker, Discrete Mathematics for Computer Scientists and Mathematicians, Prentice-Hall of India,1986.
[10] Jorgen, B.J. and Gregory, G., Digraph Theory, Algorithms and Applications, Spinger-Verlag, Berlin Heidelberg New-York, 2007.
[11] Kenneth H Rosen, Discrete Mathematics and Its Applications(with Combinatorics and Graph theory), Tata McGraw Hill, 2007.
[12] Krishnaiyan Thulasiraman, Circuit Theory Section Editor(Five Chapters), Hand book of Electronic Engineering, Academic press, oct-2000.
[13] Mac Lane. S., Categories For The Working MatheMatician, 2nd Eddition, GTM#5, Springer NY, 1998.
[14] Narasingh Deo, Graph Theory With Applications to engineering and Computer Science, Prentice Hall of India, 1990.
[15] Nistala V.E.S. Murthy and Lokavarapu Sujatha, Bresets, Communicated.
[16] Nistala V.E.S. Murthy and Lokavarapu Sujatha, Products And Factors Of Bresets, Communicated.
[17] Sambuc, R., Fonctions floues, Application l'aide au diagnostic en pathologie thyroidienne, Ph.D. Thesis University Marseille, France, 1975.
[18] Seifter, N., On the action of nilpotent and metabelian groups on infinite, locally finite graphs, European J. Combin 10 (1989) 41-45.
[19] Soardi, P.M., and woess, W., Amenability, Unimodularity, and the spectral radious of random walks on infinite graphs, Math.Z. 205 (1990) 471-486.
[20] Szasz, G., An Introduction to Lattice Theory, Academic Press, New York.
[21] Trembly, J.P., and Manohar, R., Discrete Mathematical Structures with Applications to Computer Science, McGraw Hill, 1975.
[22] Woess, W., Amenable group actions on infinite graphs, Math. Ann. 284 (1989) 251-265.

BIOGRAPHY

Nistala V.E.S. Murthy received his M.S. in Mathematics from University of Hyderabad, Hyderabad, A.P. State, India, in 1981, Ph.D. from the University of Toledo, Toledo, OH-43606, U.S.A. in 1989 and an M.S. in Software Systems from B.I.T.S., Pilani Rajasthan State, India, in 2000; taught a wide range of courses both in Mathematics and Computer Science in various levels and is currently professor in the department of Mathematics, AUCST, Andhra University, Vizag-53003, A.P. State, India. His areas of publications include Various Fuzzy Set Theories and Their Applications in Mathematics (Set Theory, Algebra and Topology) and Computer Science (Data Security/Warehousing/Mining/Hiding)

ISSN: 2350-0328

International Journal of Advanced Research in Science, Engineering and Technology

Vol. 1, Issue 5 , December 2014

and Natural Language Modeling (Reprints/Preprints are Available on Request at drnvesmurthy@rediffmail.com or at http://andhrauniversity.academia.edu /NistalaVES Murthy). In his little own way, he (1) developed f-Set Theory generalizing L-fuzzy set Theory of Goguen which generalized the [0,1]-fuzzy set theory of Zadeh, the Father of Fuzzy Set Theory (2) imposed and studied algebraic /topological structures on f-sets (3) proved Representation Theorems for f-Algebraic and f-Topological objects in general.

Lokavarapu Sujatha received her M.Sc. in mathematics from Andhra University in 2008. Currently she is working on her doctoral thesis in Lattice (algebraic) properties of representations of type 2 fuzzy sets.